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Analysis of nonlinear resistive networks having multiple solutions with spline function techniques

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Analysis of nonlinear resistive networks having multiple solutions with spline function techniques

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I. INTRODUCTION

A. Motivation for the Research

The impact of computers on circuit analysis and design and the advent of new electronic devices and integrated circuits have generated renewed interest in nonlinear circuit theory. One of the most important and basic problems in the analysis of nonlinear circuits is to determine the dc solutions (operating points) of resistive nonlinear networks containing two-terminal linear resistors, nonlinear resistors that are characterized by v-i curves or a set of measured data points, independent voltage and current sources, and the four types of the linear controlled sources. Nonlinear circuits differ from linear circuits in that they may have no solutions, an infinite number of solutions, a unique solution or several solutions. Research in nonlinear resistive networks is important not only because it is the key to the development of the analysis and design of integrated circuits but also because it serves as a prerequisite to the understanding of general nonlinear circuits and systems. A mastery of the basic concepts in nonlinear resistive networks is also crucial to a complete understanding of dynamic networks. The study of nonlinear resistive networks therefore serves as means to an end rather than the end itself.

For nonlinear resistive networks, new considerations are needed. A systematic method of formulation of the network equations must be followed, leaving less choice in circuit variables and methods of analysis than in linear network analysis. New methods of numerical solution must be developed, replacing the conventional techniques for linear systems. Unfortunately, for the more complicated nonlinear resistive networks, it is difficult to determine the number of solutions that the network has; furthermore, there exists no general and practical method of obtaining analytic solutions.

The advent of the high-speed digital computer has made the use of numerical methods for solving system of nonlinear equations not only feasible but also attractive. The numerical computation of the solutions of a nonlinear resistive network has been of considerable interest recently among network theorists and designers. Various methods are available and many computer programs exist. All the existing available methods and programs can be roughly classified into two principal groups. One is based on an iterative algorithm which is applied directly to the nonlinear network equations. The most familiar one is the Newton-Raphson method. The other is based on the piecewise-linear approximation. One limitation in the Newton-Raphson method is that, if the initial guess is not close to the true solution, the algorithm may not converge. The other limitation is that, even if it converges to a solution, there is no way of knowing whether it is the only solution or whether it converges to a particular solution, assuming that the equation has multiple solutions. This limitation is inherent in all known iteration methods. In the piecewise-linear algorithm there are also serious drawbacks. One is that at each break point a linear piecewise function is not differentiable. Furthermore most v-i curves are highly nonlinear; a higher order interpolation is always necessary for improving accuracy. Existing algorithms for obtaining multiple solutions are restricted to use for simple v-i curves. It is difficult, if not impossible, to handle networks with elements that are characterized by self-intersecting unicursal v-i curves.

B. Research Objectives

Due to the limitations and drawbacks of the existing methods, a new technique, called the piecewise-cubic spline method, is developed in this dissertation for analyzing nonlinear resistive networks with multiple solutions. The main idea is that the v-i characteristic of the nonlinear resistor is interpolated by smooth piecewise cubic functions instead of piecewise linear functions. The method of using piecewise cubic spline approximations has its main advantage over other techniques when the nonlinear element itself has corners or relatively sharp bends in the v-i curves. Thus the conditions of continuous first and/or second derivatives are met at the boundary.

Spline functions are well-developed in the literature. Their superior properties for approximation are well-recognized. In case the v-i curve is a unicursal curve with self-intersections, this technique is extremely powerful and yet straightforward in obtaining the multiple solutions. Its beauty lies in the fact that it converts a multiple solutions problem into a single solution problem.

In analyzing the network each two-terminal nonlinear resistor is replaced by its iterative piecewise-cubic spline equivalent circuit. This results in a simpler nonlinear single solution network and any Newton-like method can be applied to obtain the solution. The final numerical solution is obtained by an iterative process of substitution and verification. The new procedure always converges so long as enough iteration steps are taken.

The number of iterations required depends heavily on the proximity of the initial guess to the correct solution. But this brings no problem since each spline segment approximates a smooth portion of the original curve and there are no sharp turns.

C. Dissertation Outline

In Chapter II some of the terms and definitions are introduced for the purpose of clarifying the meaning of terms that are pertinent for the understanding of the material found in the following chapters. Chapter III contains a literature review on nonlinear resistive networks with both unique and multiple solutions. The development of spline functions is introduced and some applications in engineering fields are listed. Chapter IV deals with the mathematical preliminaries, which include the general approximation theory, approximation by polynomials and spline functions. In Chapter V two algorithms are described for the computation of piecewise spline functions with both fixed and variable knots. Chapter VI considers the procedures used for the application of spline function techniques in the analysis of nonlinear resistive networks. Two methods of analysis are discussed, nodal analysis and hybrid analysis. Chapter VII presents 6 illustrative examples in which the spline function technique is applied to the analysis of various nonlinear resistive networks.

II. TERMS AND DEFINITIONS

In this chapter some of the terms and definitions that are pertinent for the understanding of the material found in the following chapters are introduced. The definitions that the author gives are those of the references cited. They are thought to be the most standard representation.

1. [1] A spline is a mechanical device, used by draftsmen to draw a smooth curve, consisting of a strip or rod of some flexible material to which weights are attached, so that it can be constrained to pass through or near certain plotted points on a graph.
2. [2] A unicursal curve is a curve Γ in the x-y plane having the property that, if starting with one end of Γ (possibly at $-\infty$), it is possible to trace the entire curve in one continuous stroke without lifting the tip of the pencil from the paper and without retracing any portion of the curve.
3. [2] The operating point of a network N is a set of numbers which is a solution of network equations.
4. [2] A dc-network is a network which contains only dc sources.
5. [2] A dc-resistive network is a network which contains only dc sources, linear and nonlinear resistors.
6. [2] A piecewise-linear function is a function made up of a sequence of linear interpolation functions.
7. [2] A unicursal element is an element which is characterized by a unicursal curve.
8. [2] A voltage-controlled element is an element whose current

and voltage are related by the equation i = f(v).

9. [2] A current-controlled element is an element whose voltage and current are related by the equation v = f(i).

10. [2] A self-intersecting unicursal curve is any unicursal curve which intersects itself at some points.

11. [3] A bistable circuit is a circuit which has at least two stable equilibrium states.

12. [1] The knots of the spline function are a set of points at which the piecewise functions are joined.

III. LITERATURE SEARCH

A. Nonlinear Resistive Networks

1. Networks with unique solutions

Consider networks in which only two-terminal linear and nonlinear resistors and independent sources are present. Duffin [4] proved several basic theorems for such networks. However, Duffin restricted his attention to voltage-controlled resistors. For resistive networks containing strictly monotonically increasing voltage- or current-controlled elements, an early attempt at dealing with such networks is contained in a paper by Desoer and Katzenelson [5]. They also proved several useful theorems. A totally different type of approach to this problem is considered by Sandberg and Willson [6, 7] where necessary and sufficient conditions for the existence of a unique solution of the network equations are determined. They show that the problem of determining a solution for the network is equivalent to the problem of solving the equation

$$A F(\underline{x}) + B\underline{x} = \underline{c} \qquad (3\text{-}1)$$

where $F(\underline{x}) = \left(f_1(x_1), \ldots f_n(x_n)\right)^T$, x_i $(i = 1 \ldots n)$ are the port variables. The symbol A and B denote nxn matrices of real numbers, and $\underline{c}$ denotes a real n-vector. Willson [8] also derived some new theorems on the equations of nonlinear dc transistor networks. Fujisawa and Kuh [9] derived sufficient conditions for the existence of a unique solution of the equation $f(\underline{x}) = F(\underline{x}) + A\underline{x} = \underline{y}$ in terms of the Jacobian matrix $J(x) = \frac{\partial f}{\partial x}$. It is shown that if a set of cofactors of the Jacobian matrix satisfies a "ratio condition", the network has a unique solution.

The operating point problem of nonlinear circuits has been discussed in the book by Chua [2]. Two survey papers by Kuh and Hajj [3] and Willson [10] gave a detailed review on the developments of nonlinear network theory. Wu [11] investigated the operating point problem by using the degree of mapping. In a recent paper, Chua and Wang [12] have applied the degree theory to the analysis of a large class of resistive nonlinear networks. They studied the structure of the network equations by homotopy of odd fields. The form of network equations together with some circuit-theoretic conditions, such as eventual passivity, form a network function which is homotopic to an odd field. Most existing theorems relating to nonlinear resistive networks can be proved by this new approach. They consider that the concept of eventual passivity is much more basic than the so-called eventual increasingness. Nielsen and Willson [13], in their recent paper, show how transistor networks can be broken apart into smaller subnetworks, and deduce that the original circuit possesses a unique solution to its dc equations as a consequence of the uniqueness of the solutions to the dc equations of the subnetworks.

2. Networks with multiple solutions

As for the networks with multiple solutions there is still no practical general theory. General methods for determining the number of solutions possessed by the equations of a given network are needed. Chua [14] developed two computer algorithms using piecewise-linear methods for finding the dc solutions of resistive networks containing two-terminal linear and nonlinear resistors, independent dc voltage and current sources, and linear controlled sources. In his method all nonlinear resistors must be represented by piecewise-linear curves, which are tedious and time consuming at the beginning. Chao, Liu and Pan [15] develop a systematic search method for obtaining multiple solutions of simultaneous nonlinear equations. The method is based on numerical integration of the associated system of differential equations $\dot{f}_i = -f_i$ for i = 1, 2, ... n - 1 and $\dot{f}_n = \pm f_n$ along the space curve of intersection $f_i(x) = 0$, i = 1, 2, ... n - 1. The plus or minus sign is chosen so as to make $\{x_k\}$ move in the desired direction on ℓ. Chua and Ushida [16] formulate an algorithm for finding multiple solutions of a system of nonlinear algebraic equations. The algorithm consists of solving an associated system of first order differential equations whose independent variable may be switched from one variable to another during each integration step. Unfortunately, none of these algorithms can solve nonlinear resistive networks whose nonlinear elements are characterized by unicursal curves with self-intersections.

B. Spline Functions

Spline functions are a class of piecewise polynomial functions satisfying continuity properties only slightly less stringent than those of polynomials, and thus they are a natural generalization of polynomials. They are found to have highly desirable characteristics as approximating, interpolating and curve-fitting functions. Spline functions were first considered from a mathematical viewpoint by Schoenberg in 1946 [17] and became the object of rather intensive research in the late 1950's. In the 1960's spline functions had attracted wide attention. In October 1968 an advanced seminar [1] was held at Wisconsin Center on the campus of the University of Wisconsin, in Madison. The purpose of this seminar was to make a general survey of the most interesting and useful available information about spline functions and to instruct Army mathematicians in the more fundamental aspects of the theory and applications of spline functions. During the past decade numerous papers and books developing the theory of interpolation and approximation by spline functions [18, 1, 19, 20] have appeared. Concurrent with this theoretical development there has been considerable interest in practical algorithms [21, 22] for the computation of splines.

The intensive research is motivated by the following two facts: First, spline functions have certain mathematical properties that might well place them at the center of future developments in some areas of applied mathematics and numerical analysis. Second, they are the most successful approximating functions for practical applications so far discovered. The ordinary polynomials are inadequate in many situations. This is particularly the case when one approximates functions which arise from the physical world rather than from the mathematical world. Functions which express physical relationship are frequently of a disjointed or disassociated nature. In other words, their behavior in one region may be totally unrelated to their behavior in another region. Polynomials, along with most other mathematical functions, have just the opposite property. Their behavior in any small region determines their behavior everywhere. Splines do not suffer this handicap since they are defined piecewise; for $n \geq 3$, they represent nice, smooth curves in the physical world. The main mathematical interest of spline curves centers around their properties of interpolation and the minimization of certain norms [18] so that these interpolatory functions are, in some sense, best approximations to a given function.

The survey paper of de Figueiredo [20] made a general review of spline functions, describing some of the results and applications. Spline functions are applied in the following areas; (1) control theory, (2) optimal signal reconstruction and design, (3) simultaneous interpolation or smoothing in time- and frequency-domains, (4) pattern recognition, (5) stochastic processes and estimation theory, (6) digital filtering and simulation, (7) modeling of nonlinear solid state devices, (8) system modeling and identification.

IV. MATHEMATICAL PRELIMINARIES

A. Introduction

The problem of approximation of a real continuous function f(x) by an approximating function $P_n(x)$ requires answers to two important questions. The first is the type of approximating function to be used, and the second is how the "quality of an approximation" is to be measured. Needless to say it is very desirable for the approximating function $P_n(x)$ to be compatible with f(x). The problem of approximation must also be considered from different points of view. Sometimes experimental results, given in the form of either a curve or a table, must be used; sometimes a complicated mathematical expression must be replaced by a simpler and more easily treated form.

The general approach to the approximation problem consists of the following steps:

1. The first step is to translate the intuitive or practical problem into a mathematically precise form. This means that one must choose the approximating function. This step is the most important of all the steps toward obtaining an approximation. Poor choices at this point can make severe difficulties unavoidable, no matter how talented one may be at mathematical analysis.

2. The second step is to check the existence and uniqueness of the solution.

3. The third step is to find the "special" characteristics, if any, of the solution.

4. The final step is the computation of the solution.

B. General Approximation Theory

If we consider approximation by polynomials, let us ask ourselves what the effect will be for a fixed f(x) of increasing the degree of the approximating polynomial. This problem was originally proposed and solved by Weierstrass in 1885. If f(x) is continuous, it is possible to make the error of approximation arbitrarily small by increasing the degree of the approximating polynomial. The Weierstrass approximation theorem is stated as follows:

Theorem 1: [2, 23]

Let f(x) be a continuous curve representing a function over some interval $a \leq x \leq b$. It is always possible to find a polynomial $P_n(x)$ of sufficiently high degree that the magnitude of the discrepancy between $P_n(x)$ and f(x) is less than any prescribed positive number for all values of x within the interval (a, b).

Stated mathematically

$$\left|P_n(x) - f(x)\right| < \varepsilon \qquad (4\text{-}1)$$

where ε is any arbitrary small positive number.

Chua [2] in his book has another related theorem called combined interpolation and approximation theorem. It is stated as follows:

Theorem 2:

A polynomial $P_n(x)$ of sufficiently high degree can always be found that satisfies not only the Weierstrass approximation theorem but also the additional requirement that it pass through an arbitrarily prescribed finite set of points having distinct abscissas.

C. Approximation by Polynomials

1. Types of approximation

There are, generally, five frequently used conditions imposed on the polynomial approximation, depending on the use [24]. They are:

(1) The polynomial should approximate the given curve as closely as possible at one point a. This requirement means that the polynomial must go through the point a and that as many of its derivatives as possible must be equal to those of the given curve. This type of approximation is shown in Figure 1a. Mathematically, these conditions may be written in the form

$$P_n^{(i)}(a) - f^{(i)}(a) = 0 \qquad i = 0, 1, 2, ...n. \qquad (4\text{-}2)$$

(2) The polynomial should cross the given curve at several distinct points in the interval; no conditions are given for its behavior between these points. An approximation of this type is shown in Figure 1b. The mathematical expression for it is

$$P_n(x_i) - f(x_i) = 0 \qquad i = 0, 1, 2, ...n. \qquad (4\text{-}3)$$

(3) The coefficients of the polynomial are found from the condition that the area enclosed by the qth power of the difference $[P_n(x) - f(x)]$ should be minimum. An approximation of this type is called a mean approximation and is shown in Figure 2a for $q = 2$. The mathematical expression is

$$F = \min \int_a^b [P_n(x) - f(x)]^q \, dx \qquad (4\text{-}4)$$

(4) The polynomial is chosen so that, in the given interval, it never crosses the curves drawn at a distance $\pm\,\varepsilon$ parallel to the function being approximated. This distance diminishes as the degree of the polynomial is increased. An approximation of this type is called uniform as shown in Figure 2b and is expressed mathematically by the inequality:

$$\max \left| P_n(x) - f(x) \right| < \varepsilon_n. \tag{4-5}$$

(5) The polynomial is chosen so that its values oscillate above and below the given curve with equal maximum deviations. Its graph is similar to that shown in Figure 2b, the only difference being that the polynomial just touches the line with all its peaks. A mathematical expression for this type of approximation may be written as

$$\max \left| P_n(x) - f(x) \right| = E. \tag{4-6}$$

2. Interpolation and approximation formulas

There are several forms of interpolating polynomial and approximating formula which are used on computers.

a. Lagrange form interpolation polynomial [25] Let $x_0 x_1 \ldots\ldots x_n$ be n + 1 distinct points on the real axis and let f(x) be a real-valued function defined on some interval I = [a, b] containing these points. It is desired to construct a polynomial P(x) of degree $\leq$ n which interpolates f(x) at the points $x_0 \ldots\ldots x_n$, that is, which satisfies

$$P(x_i) = f(x_i) \qquad i = 0 \ldots\ldots n \tag{4-7}$$

The Lagrange form of the interpolating polynomial is one such poly-

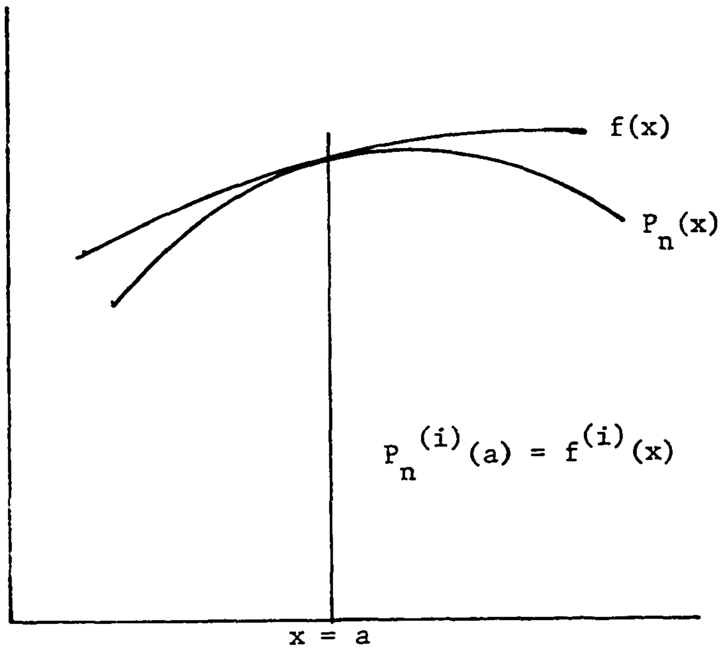


Figure 1a. Approximation at one point

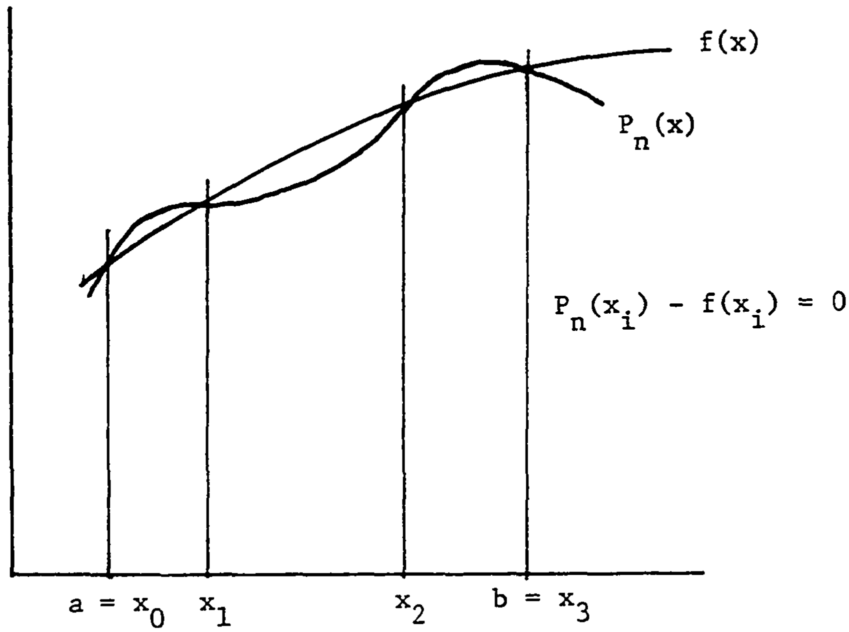


Figure 1b. Approximation at several points

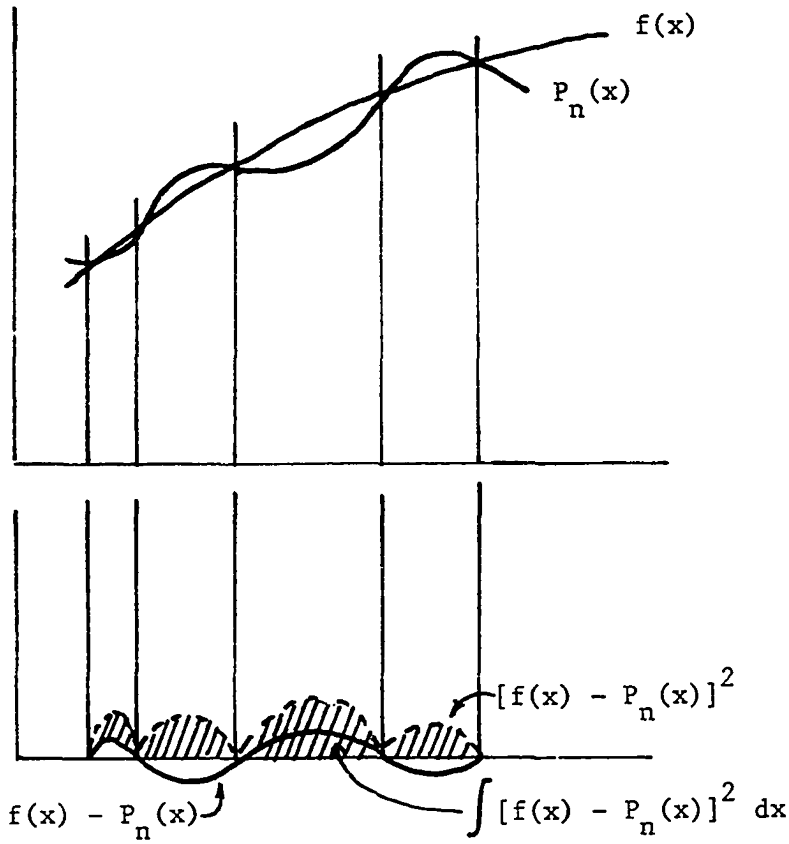


Figure 2a. Mean square approximation

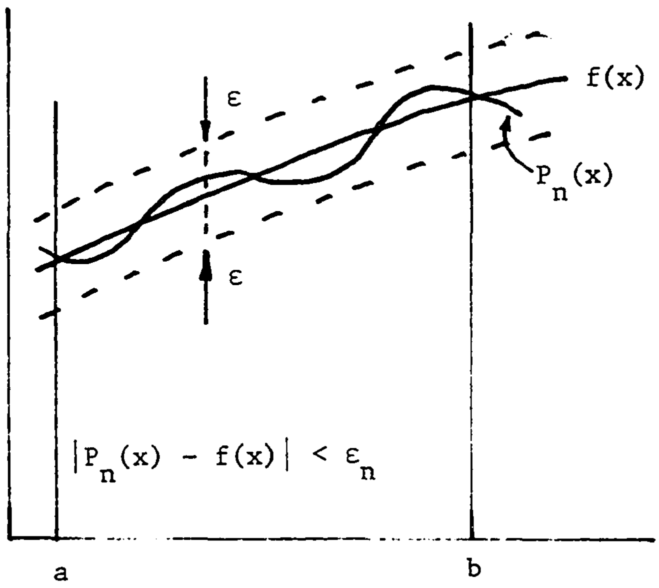


Figure 2b. Uniform approximation

nomial and can be expressed in the following form:

$$P_n(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x) \tag{4-8}$$

where $\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} (x - x_j)(x_i - x_j)^{-1}$.

For example when n = 1

$$P_1(x) = f(x_0)\ell_0(x) + f(x_1)\ell_1(x)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0). \tag{4-9}$$

The Lagrange interpolation procedure is not good because, for some analytic functions on I, the Lagrange interpolation polynomials, defined with respect to uniform meshes, diverge [26, 27]. A more serious objection to the Lagrange form arises from the uncertainty as to how many interpolation points are needed in order to find a satisfactory approximation $P_n(x)$ to f(x). Additional interpolation points require a higher degree of interpolating polynomial. Another difficulty is that the previous available information of $P_{k-1}(x)$ is not used in calculating $P_k(x)$, and this is a waste. This formula corresponds to type (2) approximation.

b. Newton form interpolation polynomial [25] The Newton form interpolation polynomial overcomes the drawbacks encountered in the Lagrange form interpolation. This form is given by the following expression.

$$P_k(x) = P_{k-1}(x) + f[x_0 \ldots x_k](x - x_0)(x - x_1) \ldots (x - x_{k-1}) \tag{4-10}$$

where $f[x_0 \ldots x_k]$ is called the kth divided difference of f(x) at the points $x_0 \ldots\ldots x_k$.

The Newton form interpolation polynomial is formally given by:

$$P_n(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$

$$+ \ldots + f[x_0, x_1, \ldots x_k](x - x_0)(x - x_1)\ldots(x - x_{k-1}). \tag{4-11}$$

In compact form it can be represented by:

$$P_n(x) = \sum_{i=0}^{n} f[x_0 \ldots x_i] \prod_{j=0}^{i-1} (x - x_j) \tag{4-12}$$

if we make use of the convention that

$$\prod_{m=r}^{s} a_m = \begin{cases} a_r a_{r+1} \ldots a_s & \text{for } r \leq s \\ 1 & \text{for } r > s \end{cases} \tag{4-13}$$

For example, when n = 1 we get that

$$P_1(x) = f[x_0] + f[x_0, x_1](x - x_0) \tag{4-14}$$

which is the same expression as Equation (4-9).

The interpolation error $e_n(x)$ of $P_n(x)$ is given by

$$e_n(x) = f(x) - P_n(x). \tag{4-15}$$

Let $\bar{x}$ be any point different from $x_0 \ldots\ldots x_n$. If $P_{n+1}(x)$ is the polynomial of degree $\leq n + 1$ which interpolates f(x) at $x_0 \ldots\ldots x_n$ and at $\bar{x}$, then $P_{n+1}(\bar{x}) = f(\bar{x})$. From Equations (4-10) and (4-12) we get

$$P_{n+1}(x) = P_n(x) + f[x_0 \ldots x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j). \quad (4\text{-}16)$$

Therefore, for all $\bar{x} \neq x_0 \ldots x_n$:

$$e_n(x) = f[x_0 \ldots x_n, \bar{x}] \prod_{j=0}^{n} (\bar{x} - x_j). \quad (4\text{-}17)$$

This formula corresponds to type (2) approximation.

c. <u>Taylor series expansion</u> The Taylor series expansion method imposes conditions on the behavior of the polynomial at one point. The approximation at this point is particularly good, but there are no means of controlling the behavior of the polynomial away from this point.

The Taylor series for f(x) about x = a is defined as

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \ldots$$

$$+ \frac{f^{(n-1)}(a)(x - a)^{n-1}}{(n-1)!} + R_n \quad (4\text{-}18)$$

$$\text{where } R_n = \frac{f^{(n)}(x_0)(x - a)^n}{n!}, \qquad a \leq x_0 \leq x, \quad (4\text{-}19)$$

is called the remainder and where it is supposed that f(x) has derivatives of order n, at least.

When the interval is sufficiently small, higher power terms can be neglected. Thus we have

$$f(x) = f(a) + f'(a)(x - a) \quad (4\text{-}20)$$

Taylor series expansion provides the analytic basis for obtaining the piecewise-linear approximation of the nonlinear v-i characteristics. This formula corresponds to type (1) approximation.

D. Piecewise Polynomial Approximation

The graphical method used in analyzing the simple nonlinear series and parallel networks is convenient. It is applicable to many practical nonlinear resistive network problems. Unfortunately, these graphical methods are not general enough to handle the more complicated non-series-parallel networks containing one or more nonlinear three-terminal resistors. Neither are they applicable to nonlinear networks containing controlled sources. Because of the above limitations, it is necessary that more general methods of analysis be found.

Generally there are two objections to polynomial approximation. The first one is that a large number of points must be used in the calculation and evaluation of the interpolating polynomial and thus it becomes costly and unreliable for large number of interpolation points. The second objection is that the higher the degree of the interpolating polynomial the greater the interpolation error [25] because interpolation error depends both on f(x) and the interpolation points.

1. Piecewise-linear interpolation

A simple and familiar example of piecewise polynomial interpolation is linear interpolation in a table of values $f(x_i)$ $i = 1\ldots n + 1$, where

$a = x_1 < x_2 \ldots\ldots < x_{n+1} = b$. Here $f(x)$ is approximated at a point $\bar{x}$ by locating the interval $[x_k, x_{k+1}]$ which contains $\bar{x}$ and then taking

$$P_1(\bar{x}) = f(x_k) + f[x_k, x_{k+1}](\bar{x} - x_k) \tag{4-21}$$

as the approximation to $f(\bar{x})$. $f(x)$ is approximated by "broken line" or "piecewise-linear function" $g_1(x)$ with break points $x_2 \ldots\ldots x_n$ which interpolates $f(x)$ at $x_1 \ldots\ldots x_{n+1}$.

At present the piecewise-linear method [23] is the only practical method of analysis for nonlinear resistive networks with multiple solutions. Though this method can be applied to any nonlinear network if sufficiently small intervals are taken, it has several drawbacks. It is not efficient as far as computer time is concerned and also has difficulties when dealing with the highly nonlinear unicursal v-i characteristics curves with self-intersections.

2. Piecewise-cubic interpolation [25, 1]

The piecewise polynomial interpolation overcomes many of the difficulties associated with piecewise linear approximation. The concept of piecewise-polynomial interpolation is first to partition the predetermined interval into subintervals and then approximate $f(x)$ in each subinterval by a suitable polynomial. With smaller intervals the interpolating error becomes small.

A piecewise-polynomial function $P_m(x)$ of degree $m > 1$ can produce approximations to $f(x)$ whose errors are much smaller than those of piecewise-linear interpolation. A special name of "spline" or "spline function" is given to the interpolant $P_3(x)$ (use $S(x)$ hereafter) whose graph approximates the position that a draftman's spline (i.e., a thin flexible rod) would occupy if it were constrained to pass through the points $\{x_i, f_i\}$, i = 1.....n+1. The draftman's spline is shown in Figure 3. A formal definition for a spline function is given below.

DEFINITION:

Given a strictly increasing sequence of real numbers, $x_1 x_2 \ldots x_n$ a spline function S(x) of degree m with the knots $x_1 x_2 \ldots x_n$ is a function defined on the entire real line having the following two properties:

(1) In each interval (x_i, x_{i+1}) for i = 0, 1, ... n (where $x = -\infty$ and $x_{n+1} = \infty$), S(x) is given by some polynomial of degree m or less.

(2) S(x) and its derivatives of orders 1, 2, ...m-1 are continuous everywhere.

Thus, a spline function is a piecewise polynomial function satisfying certain conditions regarding continuity of the function and its derivatives. In general S(x) is given by different polynomials in adjoining intervals (x_{i-1}, x_i) and (x_i, x_{i+1}). For $m > 0$, a spline function of degree m could equally well be defined as a function C^{m-1} whose mth derivative is a step function.

As mentioned before we can always interpolate a given function at four points by a cubic polynomial. Different interpolation methods differ only in how the end conditions are specified. The general form of piecewise-cubic function in interval $[x_i, x_{i+1}]$ is

$$S_i(x) = C_{1,i} + C_{2,i}(x - x_i) + C_{3,i}(x - x_i)^2 + C_{4,i}(x - x_i)^3 \tag{4-22}$$

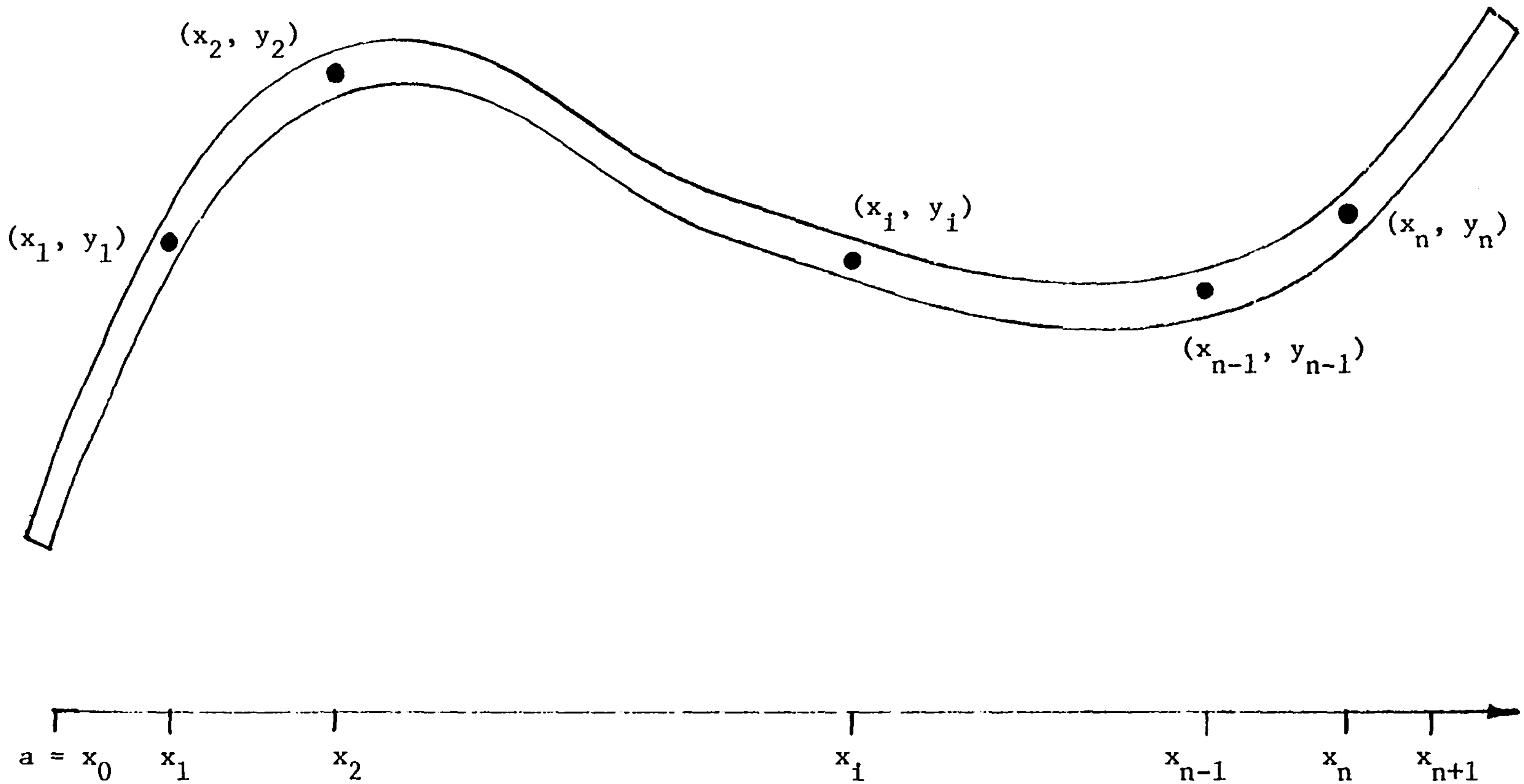


Figure 3. Draftman's spline

where $C_{j,i}$ $j = 1 \ldots 4$ are the coefficients of the spline function.

For a given set of data points

$$(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots\ldots\ldots (x_n, y_n)$$

let $a \leq x_1 < x_2 \ldots\ldots\ldots < x_n \leq b$ holds.

Let I_a and I_b be two sets of nonnegative integers, each contained in the set $\{1, \ldots\ldots m-1\}$, where $m \leq n$.

We shall require S to satisfy the conditions

$$S(x_i) = y_i \qquad (i = 1, 2, \ldots n)$$

$$S^{(r)}(a) = y_a^{(r)} \qquad (r \in I_a) \qquad (4\text{-}23)$$

$$S^{(r)}(b) = y_b^{(r)} \qquad (r \in I_b)$$

a. <u>Piecewise-cubic Hermite interpolation</u> In piecewise-cubic Hermite interpolation, one determines $S_i(x)$ so as to interpolate f(x) at $x_i, x_i, x_{i+1}, x_{i+1}$ with respect to mesh $a \leq x_1 < x_2 \ldots\ldots x_n \leq b$ and to satisfy

$$S(x_i) = f(x_i) \qquad (i = 1, \ldots n)$$

$$S_i'(x_i) = f'(x_i) \qquad (4\text{-}24)$$

$$S_i'(x_{i+1}) = f'(x_{i+1})$$

Hermite interpolation procedure is fourth order accurate [25, 28] and is a C^1-function. Piecewise-cubic Hermite interpolation requires knowledge of $f'(x)$ $i = 1, \ldots\ldots n + 1$.

b. Piecewise-cubic spline interpolation In piecewise-cubic spline interpolation one determines $S_i(x)$ so as to interpolate $f(x)$ at x_i, x_{i+1} with respect to mesh $a \leq x_1 < x_2 \ldots\ldots x_n \leq b$ and to satisfy

$$S(x_i) = f(x_i) \qquad (i = 1, \ldots n)$$

$$S'_{i-1}(x_i) = S'_i(x_i) \tag{4-25}$$

$$S''_{i-1}(x_i) = S''_i(x_i)$$

Piecewise-cubic spline interpolation procedure is also fourth order accurate [25, 28] and is a C^2-function. This procedure is an improvement over the piecewise cubic Hermite interpolation procedure in the sense that it yields a smoother interpolate. Moreover, the spline interpolate depends on roughly half as many parameters as the piecewise cubic Hermite interpolate [28].

V. SOME ALGORITHMS FOR SPLINES

Numerous papers have appeared during the last decade for the development of the theory of interpolation and approximation by spline functions. Concurrent with this theoretical development, there has been considerable interest in practical algorithms for the computation of splines.

A. Least Squares Approximation by Cubic Splines - Fixed Knots [1]

The algorithm for fixed knot least square approximation by cubic spline can be briefly stated as follows:

A basis for the cubic spline functions is established and then a modified Gram-Schmidt process is applied to obtain an orthonomal basis. The least squares approximation is computed from the orthonomal basis. Knots can be added to or deleted from the knot set and a new approximation computed. A new value can be given for one knot and a new approximation computed.

Consider the uniform approximation of an $f \in C[a, b]$ by the class of spline functions of degree m with k prescribed knots $a = x_0 < x_1 < \ldots < x_k < x_{k+1} = b$ defined by

$$\mathcal{S}_{m,k}(x_1 \ldots x_k) = \left(S(x) \in C^{m-1}[a, b] \,|\, S(x) \in \pi_m \text{ in each of the intervals } (x_i, x_{i+1}),\ i = 0, 1, \ldots, k \right) \quad (5\text{-}1)$$

Here π_m denotes the class of polynomial of degree at most m.

With the above notations the fixed knot least squares approximation problem can be stated mathematically as follows:

Given $f \in C[a, b]$ determine $S^* \in \mathcal{S}_{m,k}(x_1, \ldots x_k)$ such that

$$\left|\left|f - S^*\right|\right|_2 = \int_a^b [f(x) - S^*(x)]^2 \, dx \leq \left|\left|f - S\right|\right|_2 \qquad (5\text{-}2)$$

for every $S \in \mathcal{S}_{m,k}(x_1 \ldots x_k)$.

B. Least Squares Approximation by Cubic Splines - Variable Knots [29, 1]

This algorithm uses the fixed knot algorithm in order to determine the optimal knot locations. Each knot is, in turn, varied so as to minimize the least squares error as a function of this knot. This process is started with the right most interior knot and proceeds sequentially to the left. Iteration continues until a termination criterion is met.

Stated mathematically:

Let $\mathcal{S}_{m,k}$ be the class of splines of degree m with some k knots (allowing multiplicities) defined as before.

Given $f \in C[a, b]$ determine $S^* \in \mathcal{S}_{m,k}(x_1, \ldots x_k)$ such that

$$\left|\left|f - S^*\right|\right|_2 = \int_a^b [f(x) - S^*(x)]^2 \, dx \leq \left|\left|f - S\right|\right|_2 \qquad (5\text{-}3)$$

for every $S \in \mathcal{S}_{m,k}(x_1, \ldots\ldots x_k)$.

VI. PROCEDURES FOR THE APPLICATION OF SPLINE FUNCTIONS IN THE ANALYSIS OF NONLINEAR RESISTIVE NETWORKS

A. Approximation of Two-Terminal Nonlinear Resistor by Piecewise Cubic Spline Functions

In nonlinear resistive networks the v-i characteristic of the nonlinear resistors may be represented either by an analytic function or a set of experimental data points. The following are the procedures taken to approximate the v-i curve by piecewise-cubic spline functions.

1. Current-voltage characteristic is represented by an analytic function

1. Select proper knots: Since the successful use of splines for the purpose of providing a smooth approximation to a given set of points depends strongly on the placement of knots, this step is the most important. The number of knots is determined by the shape and the complexity of the curve. The curves between the knots are generally smooth. Thus the knots are generally selected near the maximum and minimum points of the function f(x) so that fewer piecewise cubic spline functions are required to find the optimal knot locations.

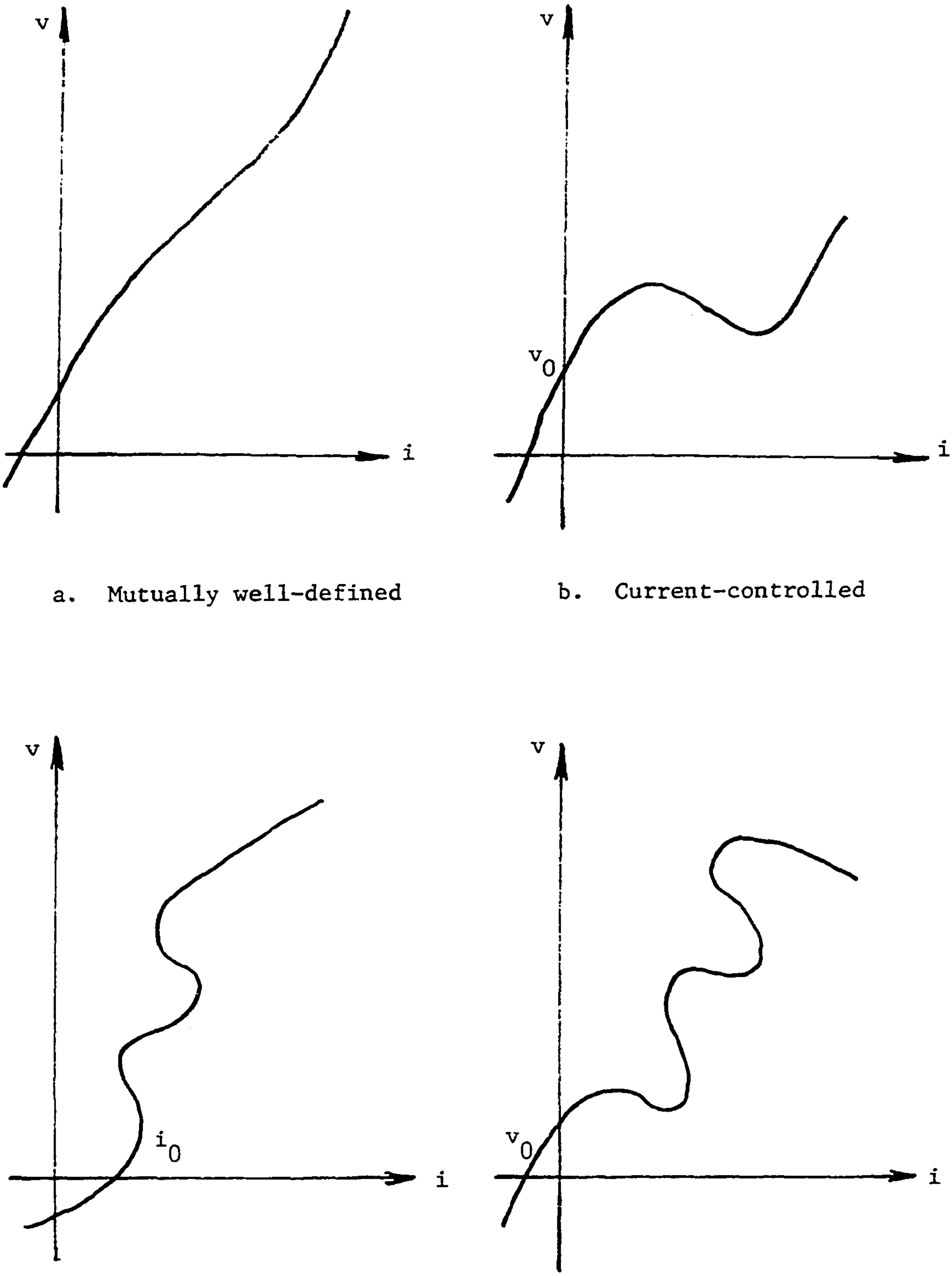
2. Calculate the corresponding $f(x_i)$ for each x_i $(i = 1....n)$

3. Calculate the coefficients $C_{i,j}$ $(i = 1...k-1,\ j = 1....4)$

4. By using Equation (4-22) and with $C_{i,j}$'s, write out the spline functions in each interval.

2. Current-voltage characteristic is in the form of an experimental curve

In general the v-i characteristic curves of two-terminal nonlinear resistors are the following two types, simple unicursal curves and unicursal curves with self-intersections as shown in Figures 4 and 5. In order to study networks containing such elements analytically, it is



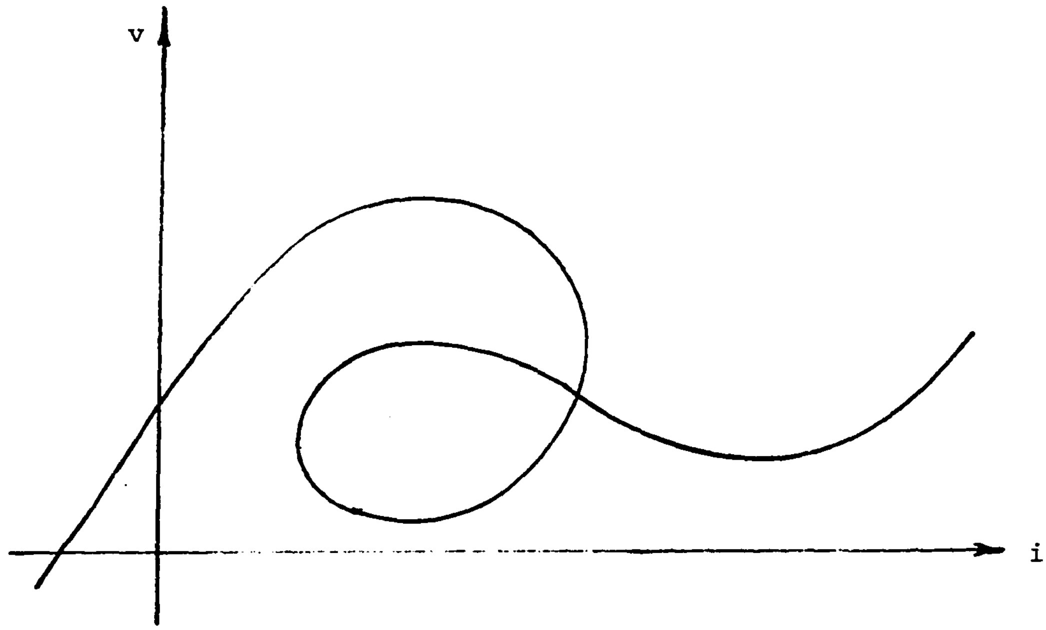
a. Mutually well-defined

b. Current-controlled

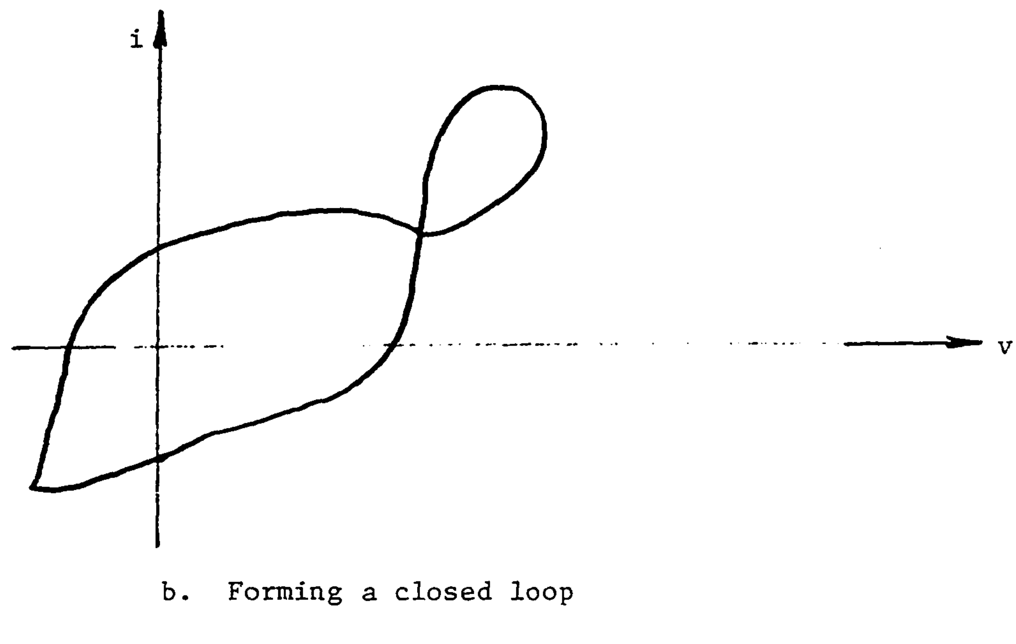
c. Voltage-controlled

d. Mutually ambiguous

Figure 4. Simple unicursal curve



a. Self-intersection



b. Forming a closed loop

Figure 5. Unicursal curve

necessary to obtain a mathematical representation of the curves. The mutually ambiguous simple unicursal curve and the unicursal curve with self-intersections can be represented either as a voltage-controlled or current-controlled element by using the piecewise-cubic approximation technique. This gives a freedom of choice in the analysis of a network and might also avoid some difficulties caused by the topological restrictions in the network.

1. Select proper knots: In this case knots $(x_1, x_2 \ldots x_k)$ are properly selected according to the shape of the curve, usually at the points where the curve is bending or changing directions. Care must be taken in case of the unicursal curve with self-intersections. In approximating this type of curve by piecewise cubic functions the curve is divided into several sections, each of which must be a functional curve. The number of sections is determined mainly by the complexity of the curve. Since the knots selected must be ordered, each section has its own sequence of knots. Of course adjacent sections share one common knot.

2. Obtain the corresponding y_i's for each x_i: This is obtained directly from the curve or from the table given.

3. Calculate the coefficients C_{ij}'s of the piecewise spline functions.

4. By using Equation (4-22) and with the C_{ij}'s, write out the spline functions in each interval.

B. Analysis of Nonlinear Resistive Networks by Piecewise Spline Function Techniques

The analysis of nonlinear resistive networks consists of two independent problems: (1) formulating the nonlinear equilibrium equations with the help of topological formulas, and (2) solving these nonlinear equations by appropriate numerical techniques. It is assumed that the nonlinear resistive networks to be analyzed have already been transformed to one containing only 2-terminal resistors. Due to the different characteristics of elements two types of nonlinear network analysis methods currently exist, nodal analysis and hybrid analysis.

1. Nodal nonlinear network analysis by spline function techniques

This type of analysis is suitable for the network consisting of linear resistors, voltage-controlled nonlinear resistors, independent sources, and linear or nonlinear voltage-controlled current sources. This restriction will allow a straightforward generalization of the nodal equation for computer solution.

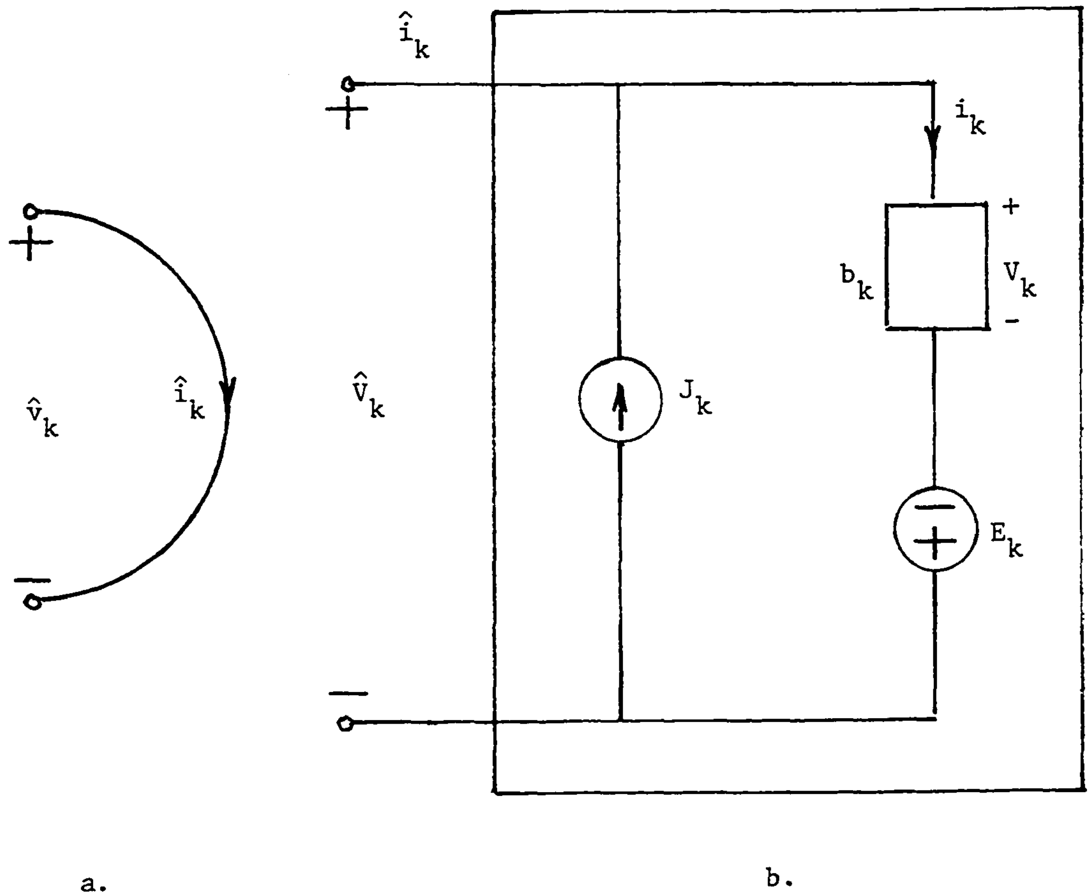
Topological formulation of nodal equations: Consider each branch "k" as shown in Figure 6a of a network graph as representing a composite branch made up of a two-terminal element b_k, an independent voltage source with terminal voltage E_k, and an independent current source with terminal current J_k, as shown in Figure 6b. The two-terminal element b_k can be either a voltage-controlled nonlinear resistor

$$i_k = g_k(v_k) \tag{6-1}$$

or a voltage-controlled current source characterized by

$$i_k = g_k(v_j) \tag{6-2}$$

where v_j is the voltage across the controlling element b_j. Equations



a. Graph for composite branch k
b. Composite branch k
Figure 6. Composite branch of nonlinear resistive network

(6-1) and (6-2) can be combined into the following compact vector form:

$$\underline{i} = \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ i_b \end{bmatrix} = \begin{bmatrix} g_1(v_\alpha) \\ g_2(v_\beta) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ g_b(v_\rho) \end{bmatrix} \triangleq \underline{g}(\underline{v}) \tag{6-3}$$

where v_α, v_β, ... v_ρ may be any branch voltage v_1, v_2, ... v_b.
From the composite branch,

$$\hat{\underline{v}} = \underline{v} - \underline{E} \tag{6-4}$$

$$\hat{\underline{i}} = \underline{i} - \underline{J} \tag{6-5}$$

From Kirchhoff's current law (KCL),

$$A\hat{\underline{i}} = 0, \tag{6-6}$$

where A is an n x b reduced incidence matrix obtained from the complete incidence matrix A_a by deleting the row corresponding to the datum node.

Substituting Equation (6-5) into Equation (6-6) gives

$$A\underline{i} = A\ \underline{J}. \tag{6-7}$$

Substituting Equation (6-3) for $\underline{i}$ in Equation (6-7), gives

$$A\underline{g}(\underline{v}) = A\ \underline{J} \tag{6-8}$$

Substituting $\underline{v}$ from Equation (6-4) into Equation (6-8)

$$Ag0(\hat{\underline{v}} + \underline{E}) = A\,\underline{J} \tag{6-9}$$

The symbol "0" is called the composition operation; whenever it appears, the quantity to its right is to be interpreted as the argument of the function appearing on its left.

Let $$\hat{\underline{v}} = A^t \underline{v}_n \tag{6-10}$$

where $\underline{v}_n$ is the node-to-datum voltage vector. Substitute Equation (6-10) for $\hat{\underline{v}}$ into Equation (6-9) to obtain the following nodal equations for nonlinear resistive networks

$$Ag0(A^t \underline{v}_n + \underline{E}) = A\,\underline{J}. \tag{6-11}$$

For an (n + 1)-node network, Equation (6-11) represents a system of n nonlinear nodal equations in term of the n node-to-datum nodal voltages. If we denote the vector $\underline{v}_n$ by $\underline{x}$, Equation (6-11) can be written in the form

$$f(\underline{x}) = 0 \tag{6-12}$$

where

$$f(\underline{x}) \triangleq Ag0(A^t \underline{x} + \underline{E}) - A\,\underline{J}. \tag{6-13}$$

Once $\underline{v}_n$ is found, $\hat{\underline{v}}$ can be computed using Equation (6-10), and $\underline{v}$ and $\underline{i}$ can be computed from Equations (6-4) and (6-5).

2. Hybrid nonlinear network analysis

Nodal analysis does not apply to networks containing current-controlled resistors or controlled voltage sources. A generalized method of analysis, called hybrid analysis, that includes both current and voltage-controlled resistors, independent sources, and all four types of linear-controlled sources, may be utilized. This method of analysis must include both currents and voltages as unknown variables - hence the name hybrid analysis. Once the hybrid equations are formulated, they can be solved by iterative techniques. In the piecewise-cubic spline technique a special procedure will be followed as will be explained later.

Let N be a nonlinear resistive network containing linear and nonlinear (both voltage- and current-controlled) resistors, constant independent voltage and current sources, and all four types of linear controlled sources. Assume that the non-monotonic voltage-controlled resistors do not form loops, and that the non-monotonic current controlled resistors do not form cutsets. If unicursal elements with mutual ambiguity are in the circuit it is always possible to avoid the above situation by proper choice of controlled variable by using piecewise spline technique. For example, if the unicursal element is in the voltage loop, consider this element as a voltage-controlled nonlinear resistor and approximate its v-i curve accordingly. Or if it forms cutsets, consider it as current-controlled nonlinear resistor.

Let a hybrid m-port $\hat{N}$ as shown in Figure 7 be formed by extracting all voltage-controlled resistors across the m_1 voltage ports and by extracting all current-controlled resistors across the m_2 current ports, where $m = m_1 + m_2$. The remaining elements in the m-port $\hat{N}$ consist only of linear resistors, constant independent sources, and linear controlled

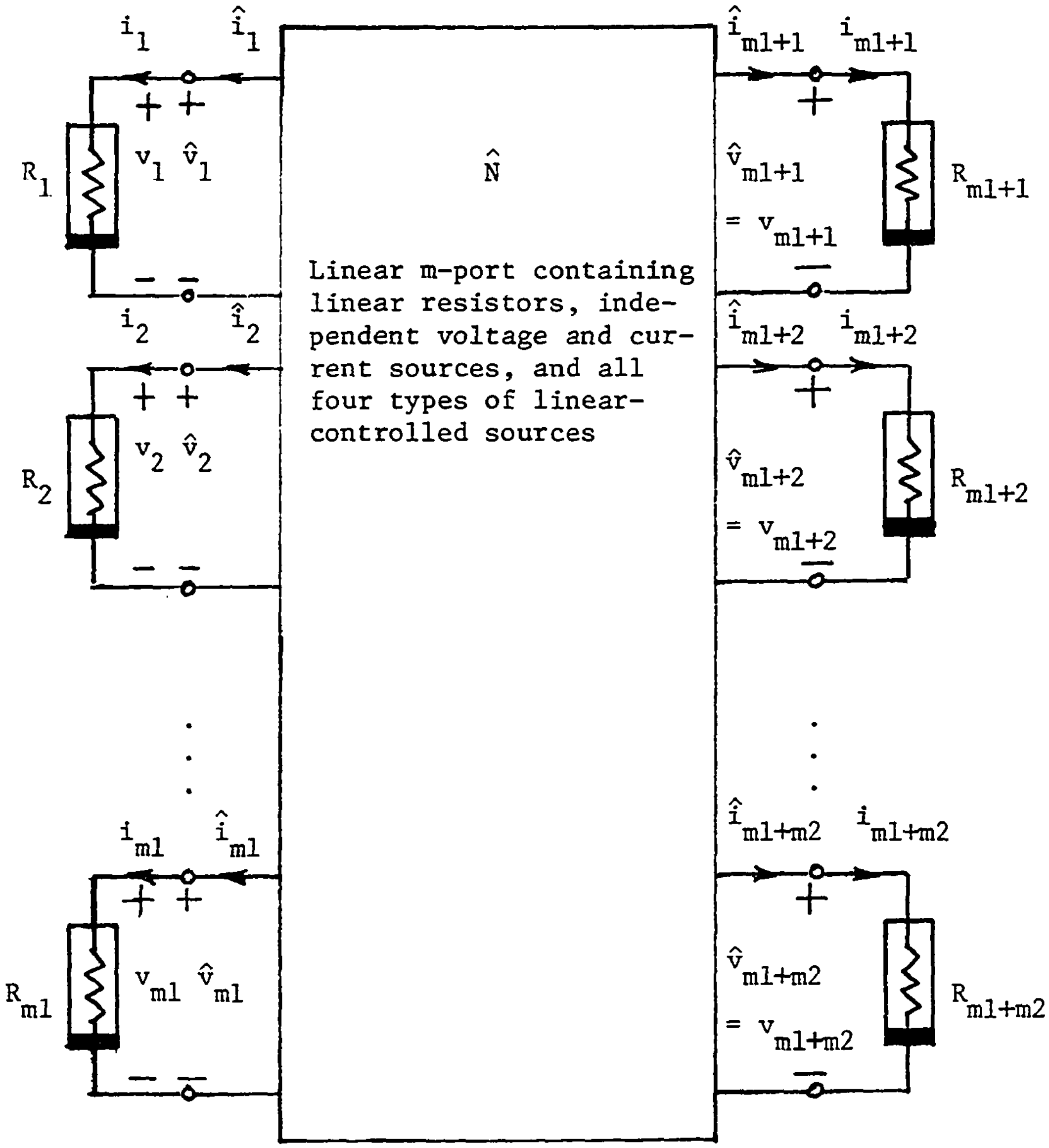


Figure 7. Nonlinear resistors of a network $\hat{N}$ are extracted and shown connected across m=m1+m2 ports of a linear m-port

sources. There is freedom in extracting a unicursal element with mutual ambiguity as a voltage-port or a current port.

Refer to the m-port of Figure 7, where the m_1 voltage ports have been labeled as port 1 to port m_1, and the m_2 current ports have been labeled as port $m_1 + 1$ to port $m_1 + m_2$. Define the following port voltage and current vectors:

$$\underline{\hat{v}}_V \triangleq \begin{bmatrix} \hat{v}_1 \\ \hat{v}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{v}_m \end{bmatrix} \qquad \underline{\hat{v}}_I \triangleq \begin{bmatrix} \hat{v}_{m_1+1} \\ \hat{v}_{m_1+2} \\ \cdot \\ \cdot \\ \cdot \\ v_{m_1+m_2} \end{bmatrix} \tag{6-14}$$

$$\underline{\hat{i}}_V \triangleq \begin{bmatrix} \hat{i}_1 \\ \hat{i}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{i}_{m_1} \end{bmatrix} \qquad \underline{\hat{i}}_I \triangleq \begin{bmatrix} \hat{i}_{m_1+1} \\ i_{m_1+2} \\ \cdot \\ \cdot \\ \cdot \\ \hat{i}_{m_1+m_2} \end{bmatrix} \tag{6-15}$$

Let $\underline{\hat{m}}$ be the vector representing the independent sources inside $\hat{N}$. Since $\hat{N}$ contains only linear resistors, independent sources, and controlled sources, the port voltages and currents may be related by a hybrid matrix $\hat{H}$ and a source vector $\underline{\hat{S}}$ as follows.

$$\begin{bmatrix} \underline{\hat{i}}_V \\ \underline{\hat{v}}_I \end{bmatrix} = \hat{H}\begin{bmatrix} \underline{\hat{v}}_V \\ \underline{\hat{i}}_I \end{bmatrix} + \hat{M}\underline{\hat{u}} = \hat{H}\begin{bmatrix} \underline{\hat{v}}_V \\ \underline{\hat{i}}_I \end{bmatrix} + \underline{\hat{S}} \tag{6-16}$$

where $\hat{\underline{S}} = \hat{M}\,\hat{\underline{u}}$.

The matrices $\hat{H}$, $\hat{M}$, and vector $\hat{\underline{S}}$ in Equation (6-16) can be partitioned according to the dimensions of $\hat{\underline{i}}_V$ and $\hat{\underline{v}}_I$ as follows:

$$\begin{bmatrix} \hat{\underline{i}}_V \\ \hat{\underline{v}}_I \end{bmatrix} = \begin{bmatrix} \hat{H}_{VV} & \hat{H}_{VI} \\ \hat{H}_{IV} & \hat{H}_{II} \end{bmatrix} \begin{bmatrix} \hat{\underline{v}}_V \\ \hat{\underline{i}}_I \end{bmatrix} + \begin{bmatrix} \hat{M}_V \\ \hat{M}_I \end{bmatrix} \hat{\underline{m}}$$

$$= \begin{bmatrix} \hat{H}_{VV} & \hat{H}_{VI} \\ \hat{H}_{IV} & \hat{H}_{II} \end{bmatrix} \begin{bmatrix} \hat{\underline{v}}_V \\ \hat{\underline{i}}_I \end{bmatrix} + \begin{bmatrix} \hat{\underline{S}}_V \\ \hat{\underline{S}}_I \end{bmatrix} \qquad (6\text{-}17)$$

where $\hat{\underline{S}}_V = \hat{M}_V\hat{\underline{u}}$ and $\hat{\underline{S}}_I = \hat{M}_I\hat{\underline{u}}$.

To obtain the hybrid equations, the v-i curves of the voltage- and current-controlled resistors are denoted by

$$\underline{i}_V \triangleq \begin{bmatrix} i_1 \\ i_2 \\ \cdot \\ \cdot \\ \cdot \\ i_{m_1} \end{bmatrix} = \begin{bmatrix} g_1(v_1) \\ g_2(v_2) \\ \cdot \\ \cdot \\ \cdot \\ g_{m_1}(v_{m_1}) \end{bmatrix} \triangleq \underline{g}_V(\underline{v}_V) \qquad (6\text{-}18)$$

and

$$\underline{v}_I \triangleq \begin{bmatrix} v_{m_1+1} \\ v_{m_1+2} \\ \cdot \\ \cdot \\ v_{m_1+m_2} \end{bmatrix} = \begin{bmatrix} f_{m_1+1}\,(i_{m_1+1}) \\ f_{m_1+2}\,(i_{m_1+2}) \\ \cdot \\ \cdot \\ f_{m_1+m_2}\,(i_{m_1+m_2}) \end{bmatrix} \triangleq \underline{f}_I(\underline{i}_I) \qquad (6\text{-}19)$$

Substituting $\hat{\underline{i}}_V = \underline{i}_V \triangleq \underline{g}_V(\underline{v}_V)$

$\hat{\underline{v}}_I = \underline{v}_I \triangleq \underline{f}_I(\underline{i}_I)$

$\hat{v}_V = v_V$ and $\hat{i}_I = i_I$

into Equation (6-17) gives

$$\underline{g}_V(\underline{v}_V) - \hat{H}_{VV}\underline{v}_V - \hat{H}_{VI}\underline{i}_I - \hat{\underline{S}}_V = 0 \tag{6-20}$$

$$\underline{f}_I(\underline{i}_I) - \hat{H}_{IV}\underline{v}_V - \hat{H}_{II}\underline{i}_I - \hat{\underline{S}}_I = 0 \tag{6-21}$$

Equations (6-20) and (6-21) constitute $m = m_1 + m_2$ independent equations in m_1 unknown voltages, $v_1, v_2, \ldots v_{m_1}$ and m_2 unknown currents, $i_{m_1+1}, i_{m_1+2}, \ldots i_{m_1+m_2}$ and are called the hybrid equations of the nonlinear resistive network $\hat{N}$. Once these unknowns are solved, the corresponding solution for the elements inside $\hat{N}$ can be easily determined by topological relations of the circuit.

VII. ILLUSTRATIVE EXAMPLES

A. Example 1 [15]

Consider the nonlinear circuit of Figure 8a, consisting of a battery E = 1.2 V in series with a linear resistor R = 1.5 kΩ and a tunnel diode. In order to find the operating points of this network analytically, the characteristics of the tunnel diode have been approximated by the following polynomial [30].

$$i = g(v) = 17.76v - 103.79v^2 + 229.62v^3 - 226.31v^4 + 83.72\ v^5 \text{ mA} \tag{7-1}$$

The v-i characteristic of the tunnel diode is shown in Figure 9. In order to find the dc solutions of this circuit by the piecewise cubic spline method, the fifth degree polynomial has to be interpolated by piecewise cubic spline functions. Following the procedures outlined in section A, Chapter VI, the first step is to select the initial knots. This function is a fifth degree polynomial and its curve has one relative maximum and one relative minimum. Together with two end points six initial knots are selected in order to take care of the flat part of the curve. They are 0.0000, 0.1380, 0.5000, 0.6601, 0.8500 and 1.0000 volts. After using the variable knot computer subroutine ICSVKU [29] which is briefly explained in Appendix B, the optimal knots for minimum least squares error for five piecewise cubic spline functions are 0.0000, 0.1711, 0.4075, 0.4919, 0.7796, and 1.0000 volts. With these optimal knots, the Y vector, which represents the constant term $C_{i,1}$ (i = 1, ... 5), and

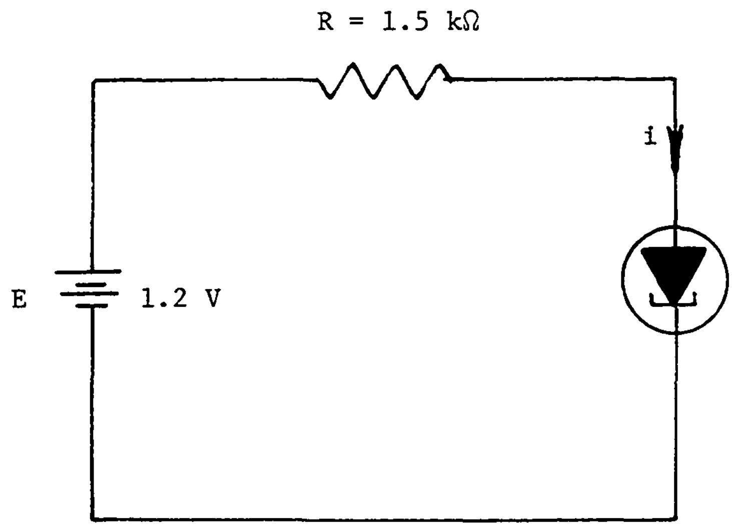


Figure 8a. Single tunnel-diode circuit

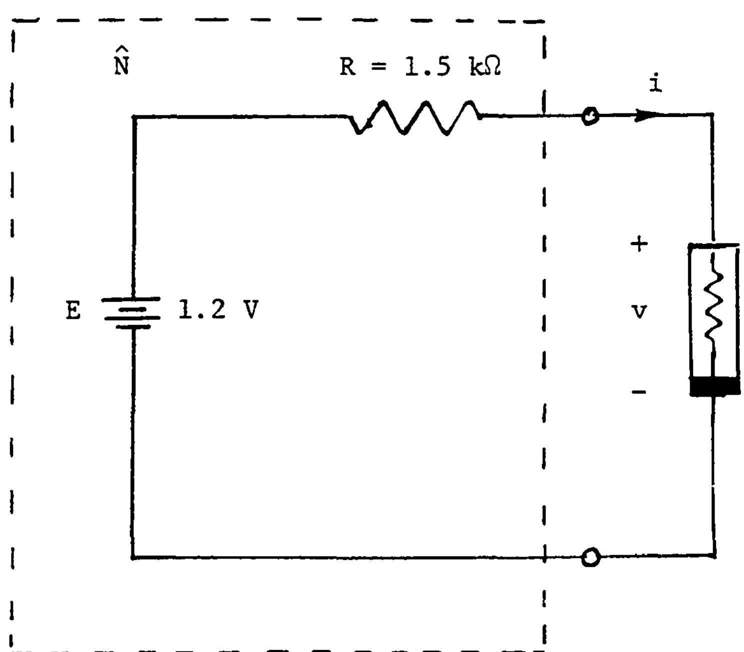


Figure 8b. Extraction of the nonlinear resistor of network $\hat{N}$

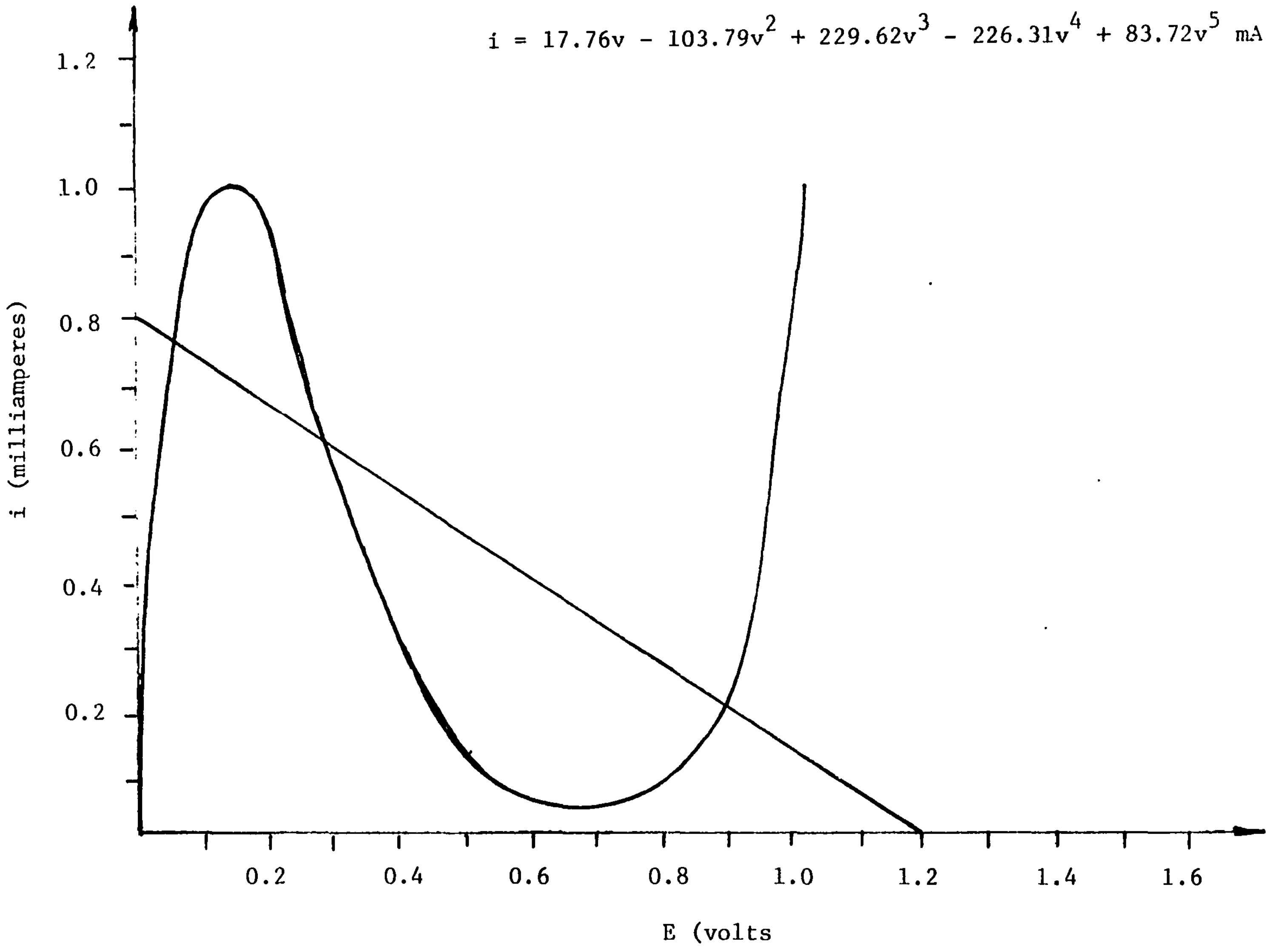


Figure 9. v-i characteristics for tunnel diode

the matrix C for coefficients $C_{i,j}$ (i = 1, ... 5, j = 2 ... 4) of cubic spline functions are as follows:

$$Y = C_{i,1} = \begin{bmatrix} 0.0004074 \\ 0.9655 \\ 0.2090 \\ 0.1106 \\ 0.08211 \end{bmatrix}$$

where $C_{i,1}$ i = 1, ... 5 is the constant term for each spline function.

$$C = C_{ij} = \begin{bmatrix} 17.47 & -95.17 & 152.2 \\ -1.734 & -17.06 & 45.92 \\ -2.101 & 15.50 & -52.37 \\ -0.6034 & 2.237 & -1.679 \\ 0.2665 & 0.7874 & 75.90 \end{bmatrix}$$

where C_{ij} i = 1, ... 5, j = 2, ... 4 are the coefficients for spline functions.

The piecewise cubic spline functions for each interval are:

S_1 $\quad 0.0004074 + 17.47\ v - 95.17\ v^2 + 152.2\ v^3$

(between knots 0.00 and 0.1711)

S_2 $\quad 0.9655 - 1.734\ (v - 0.1711) - 17.06\ (v - 0.1711)^2 + 45.92\ (v - 0.1711)^3$

(between knots 0.1711 and 0.4075)

S_3 $\quad 0.2090 - 2.101\ (v - 0.4075) + 15.5\ (v - 0.4075)^2 - 52.37\ (v - 0.4075)^3$

(between knots 0.4075 and 0.4919)

S_4 $\quad 0.1106 - 0.6034\ (v - 0.4919) + 2.237\ (v - 0.4919)^2$

$- 1.679\ (v - 0.4919)^3$

(between knots 0.4919 and 0.7796)

S_5 $\quad 0.08211 + 0.2665\ (v - 0.7996) + 0.7874\ (v - 0.7796)^2$

$+ 75.9\ (v - 0.7796)^3$

(between knots 0.7796 and 1.000)

All the spline functions have continuous first and second derivatives at the knots.

For the simple circuit of Figure 8a, the following equations can be obtained by inspection.

Equation from law of interconnection (Kirchhoff's Laws):

$$f(v,i) = v + R\ i - E = 0 \qquad (7\text{-}2)$$

Equation from law of element:

$$g(v) = 17.76\ v - 103.79\ v^2 + 229.62\ v^3 - 226.31\ v^4$$

$$+ 83.72\ v^5\ \text{mA} \qquad (7\text{-}3)$$

The network of Figure 8a is redrawn as shown in Figure 8b in the form of a one-port linear network terminated by the iterative spline equivalent circuits associated with the nonlinear resistor. S_i (i = 1, ... 5) is the iterative piecewise cubic spline. For each S_i the nonlinear spline equivalent circuit equations with initial values chosen for example, at the midpoint of the interval, are solved. The validity of each solution is checked by comparing with the interval limits. If it falls outside the interval limits, it is a false solution, otherwise it is a true solution. By applying the subroutine ZSYSTM [29] the three valid solutions (operating points) are found to be Q_1 (0.0626 V, 0.7584 mA), Q_2 (0.2875 V, 0.6046 mA), Q_3 (0.8857 V, 0.2095 mA). Subroutine ZSYSTM is briefly explained in Appendix C. In this example Q_1 is obtained with 3 iterations and Q_2 and Q_3 each with 1 iteration. Q_1, Q_2 and Q_3 are the equilibrium states of this bistable circuit. Q_1 and Q_2 are stable and Q_2 is unstable. The complete computer program is listed in Appendix D.

If we apply the subroutine ZSYSTM and solve directly the system of equations with the original fifth degree polynomial for the tunnel diode and five initial conditions selected as in the piecewise-cubic case, we obtain the following five solutions: Q_1 (0.0622 V, 0.7584 mA), Q_2 (0.0621 V, 0.7585 mA), Q_3 (0.0082 V, 0.7945 mA), Q_4 (0.1865 V, 0.6756 mA), Q_5 (0.2875 V, 0.6082 mA). By comparing these solutions with those obtained by piecewise-cubic spline techniques only solutions Q_1 and Q_5 are the true solutions and the other true solution is missing though the same initial conditions were used.

B. Example 2

Consider the same nonlinear circuit of Figure 8a. If the value of the independent voltage source E is changed while the value of the series resistor R is fixed, a family of loadlines with the same slope is obtained as shown in Figure 10. The intersections of those loadlines with the tunnel diode characteristics curve are the operating points at that particular bias condition. The locus of operating points is obtained with the help of the computer and is tabulated as shown in Table 1. In the

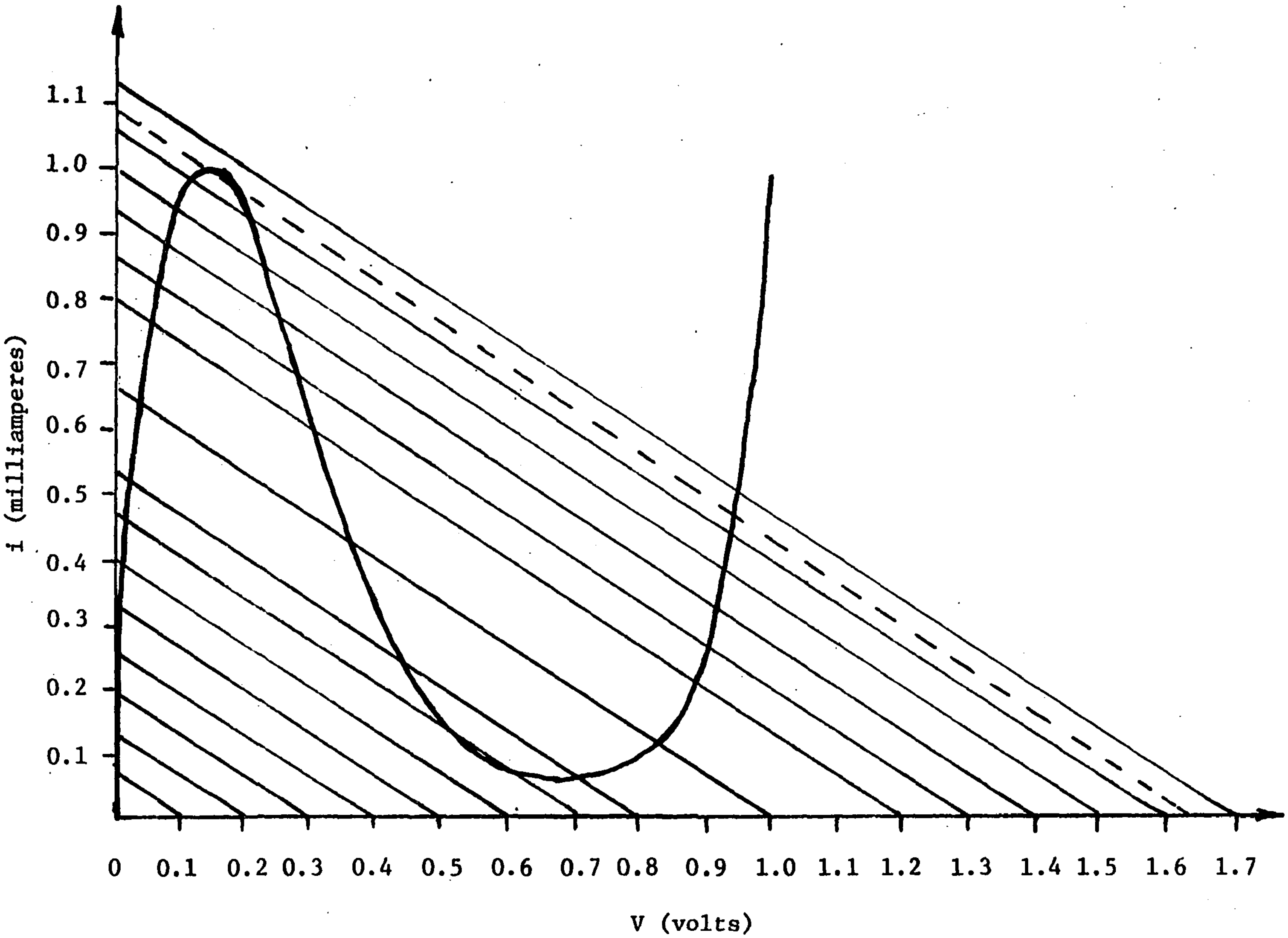


Figure 10. A family of loadlines superimposed upon the tunnel diode v-i curve

Table 1. The locus of operating points for Example 2

Value of E (volts)	Coordinates of operating points (milliamperes)									
	V_{11}	V_{12}	V_{13}	V_{14}	V_{15}	I_{11}	I_{12}	I_{13}	I_{14}	I_{15}
0.1	0.0037	-	-	-	-	0.0641	-	-	-	-
0.2	0.0076	-	-	-	-	0.1282	-	-	-	-
0.3	0.0117	-	-	-	-	0.1921	-	-	-	-
0.4	0.0159	-	-	-	-	0.2560	-	-	-	-
0.5	0.0204	-	-	-	-	0.3196	-	-	-	-
0.6	0.0252	-	-	-	-	0.3831	-	-	-	-
0.64	0.0272	-	-	-	-	0.4085	-	-	-	-
0.65	0.0277	-	-	-	-	0.4148	-	-	-	-
0.66	0.0282	-	0.4622	-	-	0.4211	-	0.1318	-	-
0.7	0.0302	-	0.4185	-	-	0.4464	-	0.1876	-	-
0.8	0.0356	0.3788	-	-	-	0.5095	0.2807	-	-	-
0.9	0.0414	0.3520	-	0.7775	-	0.5723	0.3653	-	0.0816	-
1.0	0.0478	0.3289	-	-	0.8337	0.6347	0.4473	-	-	0.1108
1.1	0.0547	0.3074	-	-	0.8645	0.6968	0.5283	-	-	0.1569
1.2	0.0625	0.2864	-	-	0.8855	0.7582	0.6090	-	-	0.2096
1.3	0.0715	0.2650	-	-	0.9019	0.8189	0.6899	-	-	0.2653
1.4	0.0823	0.2424	-	-	0.9155	0.8784	0.7117	-	-	0.3229
1.5	0.0962	0.2168	-	-	0.9276	0.9358	0.8544	-	-	0.3816
1.6	0.1168	0.1845	-	-	0.9387	0.9887	0.9436	-	-	0.4408
1.601	0.1222	-	-	-	0.9383	0.9918	-	-	-	0.4477
1.604	0.1444	-	-	-	0.9411	0.9970	-	-	-	0.4658
1.605	-	-	-	-	0.9421	-	-	-	-	0.4719
1.606	-	-	-	-	0.9430	-	-	-	-	0.4779
1.7	-	-	-	-	0.9494	-	-	-	-	0.5003
1.8	-	-	-	-	0.9597	-	-	-	-	0.5601
1.9	-	-	-	-	0.9699	-	-	-	-	0.6200
2.0	-	-	-	-	0.9800	-	-	-	-	0.6799

table V_{ij} and I_{ij} represent the voltage and current of the ith nonlinear element jth spline segment respectively. For example V_{14} and I_{14} represent the voltage and current of the first nonlinear element and 4th spline segment respectively. By changing the value of independent voltage from 0.1 V to 2.0 V each spline segment has at least one operating point, and the maximum number of operating points is three as is shown in Table 1. When the value of E is between 0.7 V and 1.604 V it provides bistable conditions. When E exceeds 1.604 V, switching occurs as is indicated by the dashed line in Figure 9. By changing the bias voltage in this simple circuit, it senses the threshold level for a switching [26, 27]. This circuit can be used as a tunnel diode switching circuit and has threshold-sensing property. Because of this property it is useful in nuclear instrumentation where the energy of particles under study is related to the amplitude of the pulses they produce.

C. Example 3

Consider the nonlinear circuit of Figure 11a, consisting of a battery E = 2.0 V in series with a linear resistor R = 5 kΩ and two tunnel diodes. The characteristics of the two tunnel diodes are the same as was used in Example 1. In this circuit we have two nonlinear elements. By applying the spline function technique as described in Example 1, each nonlinear element can be approximated by 5 piecewise-cubic spline functions. The iterative equivalent circuit is shown in Figure 11b.

The circuit equations are:

Equations from law of elements:

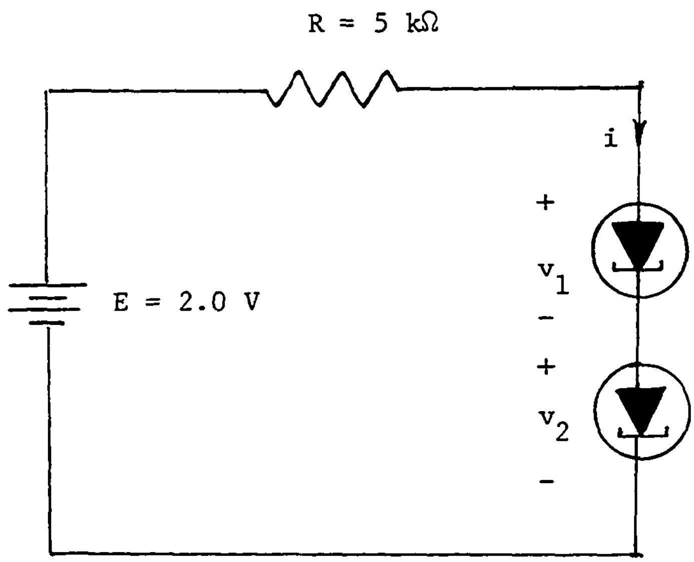


Figure 11a. Two-tunnel-diode circuit

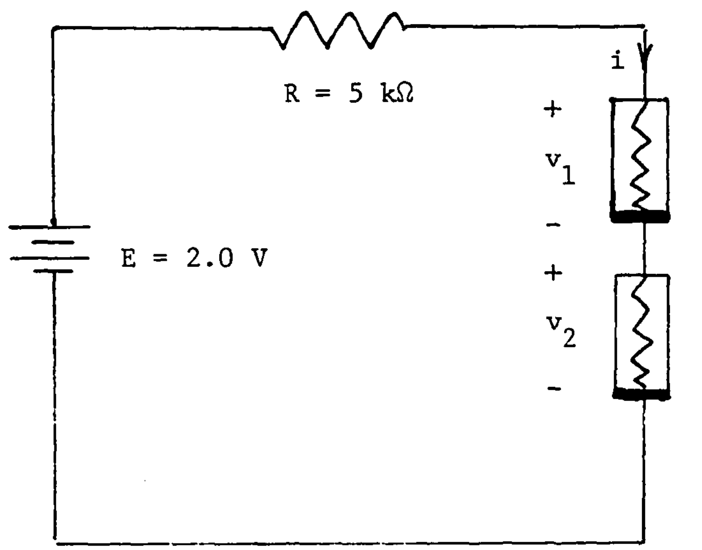


Figure 11b. Iterative equivalent circuit

$$i - g_1(v_1) = 0 \tag{7-4}$$

$$i - g_2(v_2) = 0 \tag{7-5}$$

Equation from law of interconnection:

$$v_1 + v_2 + 5i - 2 = 0 \tag{7-6}$$

To solve the system of simultaneous nonlinear equations, $g_1(v_1)$ and $g_2(v_2)$ must be substituted by their iterative equivalent piecewise spline functions. There are a total of 25 (5x5) spline segment combinations. With each segment combination the circuit can be solved easily by any Newton-like iterative method, since all the equations are in lower degree and there are no initial value and singularity problems. After the solution is obtained, its validity is checked by comparing it with the interval limits. By applying the subroutine ZSYSTM there are 9 true solutions and 16 false solutions. The maximum number of iterations for all the solutions is 6. The 9 true solutions together with their segment combinations are shown in Table 2. The characteristics of two series tunnel-diode and the loadline is shown in Figure 12. If the value of the independent voltage source E and the series resistor R are properly selected, different operating points are obtained as shown in Table 2. With E = 1.2 V and R = 1.5 kΩ, there are only 7 true solutions, the other 18 are false solutions. With E = 1.2 V and R = 2 kΩ only 6 true solutions are obtained.

D. Example 4

In the previous examples the nonlinear elements are characterized by

Table 2. The locus of operating points for twin-tunnel-diode circuit

Value of E (volts)	Value of R (kΩ)	Spline segment combination	V_1 (volts)	V_2 (volts)	I (mA)
		11	0.0257	0.0257	0.3897
		12	0.0207	0.3649	0.3228
		15	0.0135	0.8888	0.2195
		21	0.3649	0.0207	0.3228
2.0	5	22	0.3928	0.3928	0.2428
		35	0.4555	0.8542	0.1380
		51	0.8888	0.0134	0.2195
		53	0.8542	0.4555	0.1380
		55	0.7885	0.7885	0.0845
		11	0.0579	0.0579	0.7227
		12	0.0417	0.2951	0.5753
		15	0.0124	0.8833	0.2028
1.2	1.5	21	0.2951	0.0417	0.5753
		22	0.3702	0.3702	0.3063
		44	0.5328	0.5328	0.8952
		51	0.8833	0.0124	0.2028
		11	0.0402	0.0402	0.5597
		12	0.0279	0.3371	0.4174
1.2	2	15	0.0097	0.8668	0.1617
		21	0.3371	0.0279	0.4174
		33	0.4279	0.4279	0.1720
		51	0.8668	0.0097	0.1617

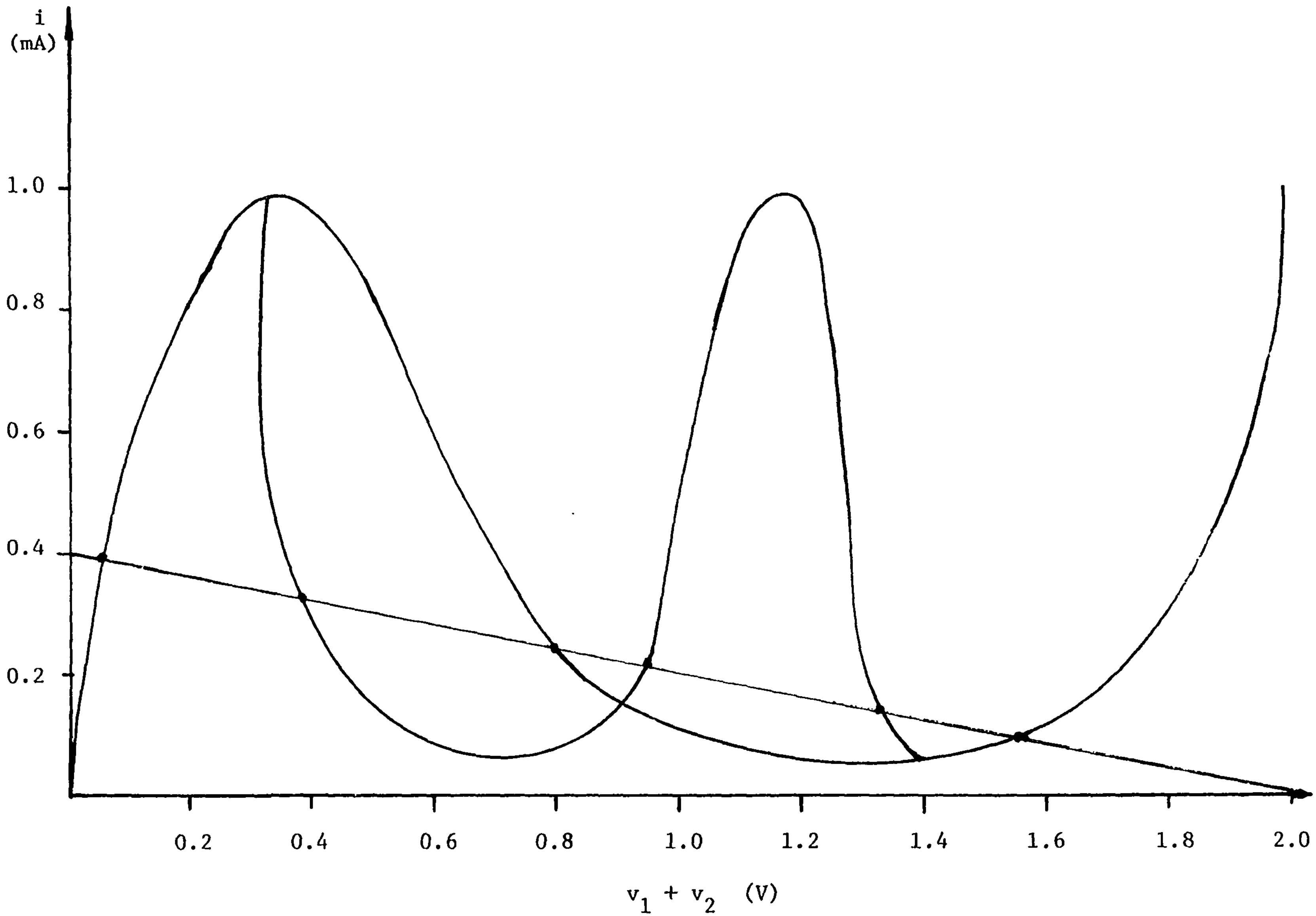


Figure 12. The characteristics of two series tunnel-diode and the loadline

simple unicursal curves. If the nonlinear element is characterized by unicursal curves with self-intersections, the piecewise spline function technique is the only existing practical method which can be used to solve the problem and obtain the multiple solutions.

Consider the same nonlinear circuit of Figure 8a. With E = 18 V and R = 2 kΩ the v-i characteristic of the nonlinear element is as shown in Figure 13, which is of course pathological. For approximation by piecewise cubic spline functions, the whole curve is divided into three sections, Γ_1, Γ_2 and Γ_3. Their intervals are Γ_1 [(0.00, 0.00), (14.6, 7.7)], Γ_2 [(5.2, 14.6), (3.0, 7.7)] and Γ_3 [(5.3, 18.0), (3.0, 5.3)]. The choice of dividing boundaries requires each of the Γ_i (i = 1, 2, 3) to be a functional curve, so that the computer subroutine to obtain the piecewise spline functions can be applied directly. For Γ_1 16 points are selected along the curve for approximation, Γ_2 11 points, and Γ_3 13 points. After using the subroutine ICSVKU for each Γ_i (i = 1, 2, 3) the computed results are as follows:

Γ_1

$$Y = \begin{bmatrix} 0.0901 \\ 7.5790 \\ 9.0280 \end{bmatrix}$$

where $Y_i = C_{i.1}$ is the constant term of spline functions and

$$C = \begin{bmatrix} 1.9660 & -0.1913 & 0.00907 \\ 0.6213 & -0.00126 & -0.00890 \\ -0.5823 & -0.1793 & 0.00138 \end{bmatrix}$$

The optimal knots are (0.000, 6.983, 13.65, 14.60) and the piecewise spline functions are:

$S_1 \quad 0.09015 + 1.966\ v - 0.1913\ v^2 + 0.00907\ v^3$

$S_2 \quad 7.579 + 0.6213\ (v - 6.983) - 0.00126\ (v - 6.983)^2$
$\qquad - 0.0089\ (v - 6.983)^3$

$S_3 \quad 9.028 - 0.5823\ (v - 13.65) - 0.1793\ (v - 13.65)^2$
$\qquad + 0.001387\ (v - 13.65)^3$

2

$$Y = \begin{bmatrix} 3.001 \\ 2.006 \\ 5.910 \end{bmatrix}$$

and

$$C = \begin{bmatrix} -1.9740 & 1.2570 & -0.2575 \\ 0.07102 & 0.02894 & 0.004954 \\ 1.2670 & 0.1365 & 5.3100 \end{bmatrix}$$

The optimal knots are (5.20, 6.79, 14.02, 14.60) and the spline functions are:

$S_1 \quad 3.001 - 1.974\ (v - 5.20) + 1.257\ (v - 5.20)^2$
$\qquad - 0.2575\ (v - 5.20)^3$

$S_2 \quad 2.006 + 0.07102\ (v - 6.79) + 0.02894\ (v - 6.79)^2$
$\qquad + 0.004954\ (v - 6.79)^3$

$$S_3 \quad 5.91 + 1.267\ (v - 14.02) + 0.1365\ (v - 14.02)^2 + 5.31\ (v - 14.02)^3$$

Γ_3

$$Y = \begin{bmatrix} 3.1830 \\ 4.6790 \\ 2.8000 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1.4000 & -0.9309 & -0.02973 \\ -1.4040 & -0.5087 & 0.2653 \\ -1.5240 & 0.4040 & -0.01998 \end{bmatrix}$$

The optimal knots are (5.20, 9.86, 11.01, 18.00) and the piecewise spline functions are:

$$S_1 \quad 3.183 + 1.4\ (v - 5.20) - 0.09309\ (v - 5.20)^2 - 0.02973\ (v - 5.20)^3$$

$$S_2 \quad 4.679 - 1.404\ (v - 9.86) - 0.5087\ (v - 9.86)^2 + 0.2653\ (v - 9.86)^3$$

$$S_3 \quad 2.8 - 1.524\ (v - 11.01) + 0.4041\ (v - 11.01)^2 - 0.01998\ (v - 11.01)^3$$

The circuit equations are:

$$f(v,i) = v + R\ i - E = 0$$

$$i - S(v) = 0$$

After applying the subroutine ZSYSTM to solve the nonlinear equations, 5 true solutions are found. These 5 operating points are the intersections of the v-i characteristics and the load line as shown in Figure 13. Their coordinates are:

Q_1 (5.1867 V, 6.4066 mA), Q_2 (7.2233 V, 5.3883 mA)

Q_3 (10.411 V, 3.7941 mA), Q_4 (11.282 V, 3.3585 mA)

Q_5 (14.6781 V, 1.6609 mA).

E. Example 5

Consider the more general nonlinear resistor network consisting of an independent voltage source E, linear resistors, a voltage-controlled resistor R_1, a current-controlled resistor R_2 and a current-controlled voltage source as shown in Figure 14. R_1 is a tunnel diode and its v-i characteristics are the same as the one used in Example 1 and R_2 is an unicursal element whose v-i characteristics are the same as the one used in Example 4 except that the roles of v and i are interchanged. In other words R_2 is now a current-controlled nonlinear resistor. If we extract two nonlinear resistors R_1 and R_2 as shown in Figure 14, then we can apply the procedure described in section B, Chapter VI for hybrid nonlinear network analysis. The hybrid representation of this circuit is

$$\begin{bmatrix} \hat{i}_1 \\ \hat{v}_2 \end{bmatrix} = \begin{bmatrix} \hat{H}_{11} & \hat{H}_{12} \\ \hat{H}_{21} & \hat{H}_{22} \end{bmatrix} \begin{bmatrix} \hat{v}_1 \\ \hat{i}_2 \end{bmatrix} + \begin{bmatrix} \hat{S}_1 \\ \hat{S}_2 \end{bmatrix} \tag{7-7}$$

For this circuit each element of $\hat{H}$ can be found [31] by first setting all independent sources inside $\hat{N}$ to zero, so that $\underline{\hat{S}} = \hat{M}\, \underline{\hat{u}} = 0$, and then obtaining $\hat{h}_{jk}$ by the ratio

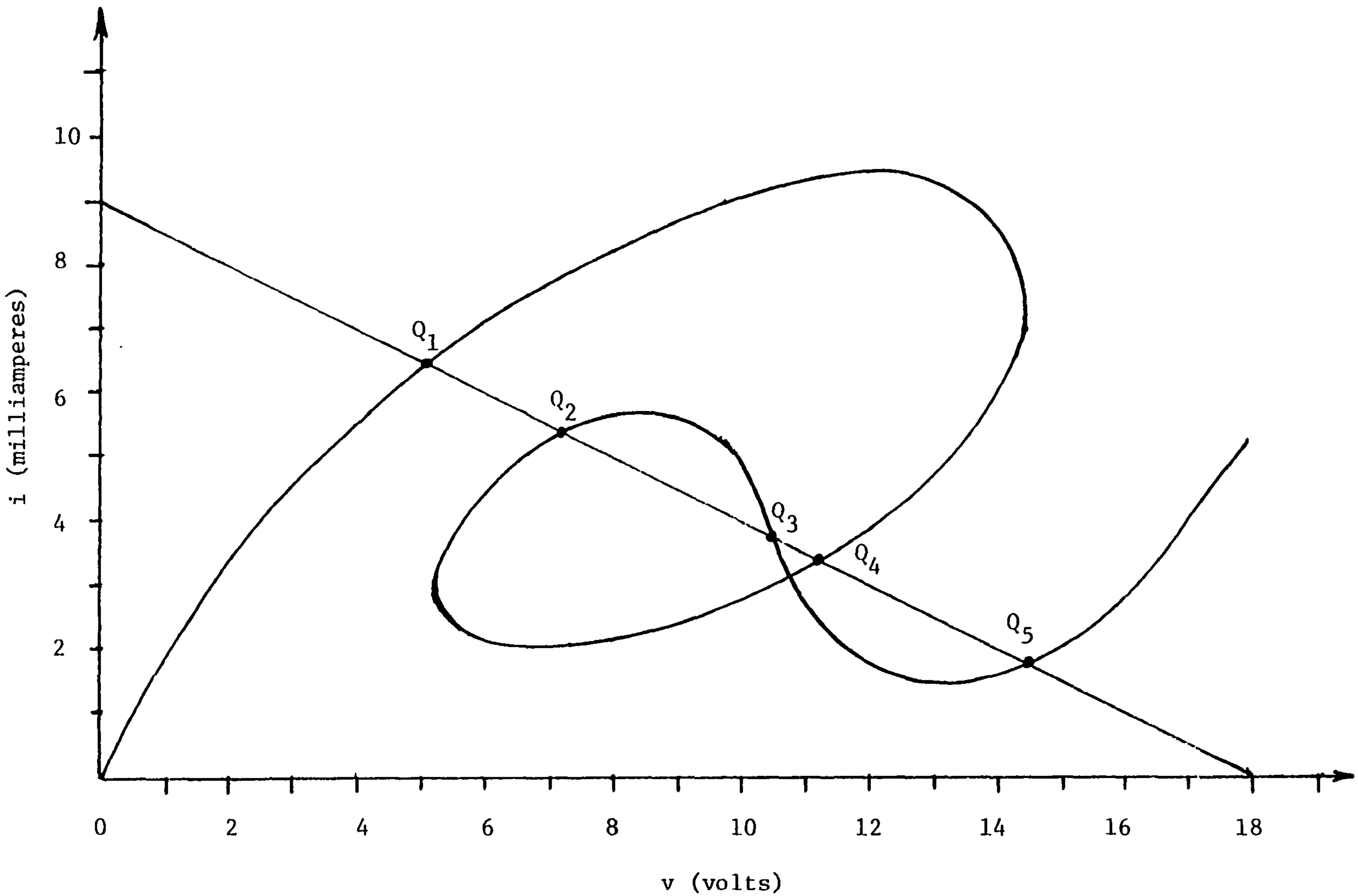


Figure 13. v-i characteristics of the unicursal element

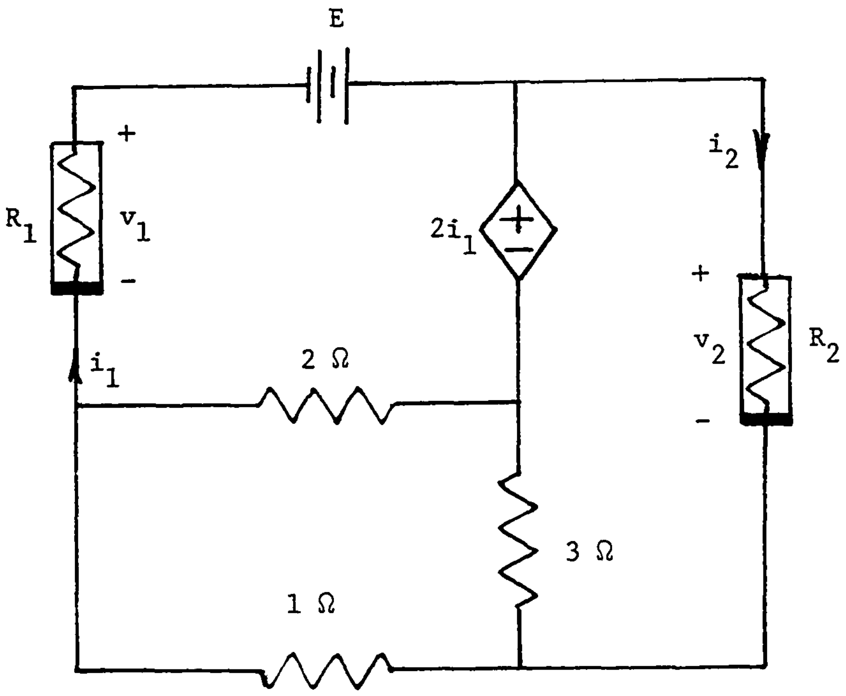


Figure 14. Circuit for Example 5

$$\hat{h}_{jk} = \frac{\text{response at port j}}{\text{excitation at port k}} \tag{7-8}$$

$\hat{S}_1$ and $\hat{S}_2$ can be found as follows:

$$\begin{aligned} \hat{S}_1 &= \hat{i}_1 \big| \hat{v}_1 = 0,\ \hat{i}_2 = 0 \\ \hat{S}_2 &= \hat{v}_2 \big| \hat{v}_1 = 0,\ \hat{i}_2 = 0 \end{aligned} \tag{7-9}$$

After some algebraic manipulations the resulting hybrid equations are found to be:

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{10} & \frac{3}{10} \\ -\frac{9}{10} & -\frac{3}{5} \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 0.36 \\ 1.08 \end{bmatrix}$$

where i_1, v_1 and i_2, v_2 are the voltage and current of the voltage and current port respectively. And

$$i_1 = g_1(v_1)$$

$$v_2 = f_2(i_2)$$

are the voltage-controlled and current-controlled resistor respectively.

Since there is no general method to determine the number of solutions of the nonlinear network with multiple solutions, several values of E are tried. With the help of computer subroutine ZSYSTM, when E = 1.2 V or 0.5 V there is only one true solution. When E = 10 V or 30 V no true solution is obtained. The dc solutions (operating points) are:

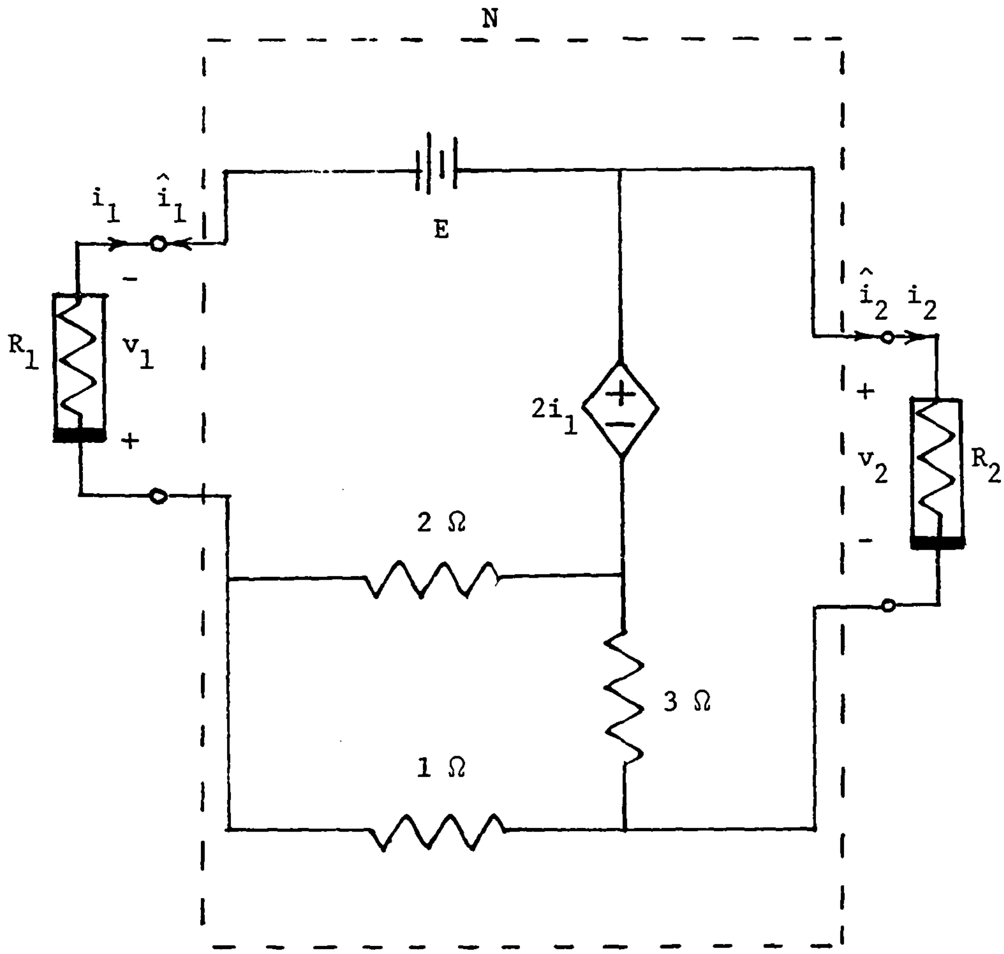


Figure 15. Circuit for Example 5. Nonlinear resistors are extracted and shown across 2 ports of a linear 2-port

$Q_1(v_1 = 0.0319\text{ V},\ i_1 = 0.4661\text{ mA},\ v_2 = 0.8198\text{ V},\ i_2 = 0.3855\text{ mA})$

when $E = 1.2$ V

$Q_2(v_1 = 0.0114\text{ V},\ i_1 = 0.1878\text{ mA},\ v_2 = 0.3571\text{ V},\ i_2 = 0.1376\text{ mA})$

when $E = 0.5$ V

F. Example 6 [30]

Tunnel diodes can be used in the flip-flop circuit. Figure 16a is a two-tunnel-diode circuit. It consists of two tunnel diodes connected in series, three resistors, and an inductor. The two tunnel diodes are assumed to be identical and their v-i characteristics are the same as the one used in Example 1. The principle of operation is as follows: the dc supply is of such a magnitude that only one of the two diodes can be in the high-voltage, low-current state; the other diode has to be in the low-voltage, high-current state.

To design this flip-flop circuit, the first step is to determine the proper values of R_b, R, and E so that TD1 is biased at point b and TD2 is biased at point a as is shown in Figure 16b. The second step is to find the operating points. In the steady state the circuit is a nonlinear dc resistor circuit, so the spline function technique can be applied to obtain the multiple solutions.

Referring to Figure 16b, the supply voltage E has to be at least large enough to let one of the diodes pass its peak current point but not large enough either to support both diodes at the valley point (V_v, I_v), or one diode at the valley point (V_V, I_V) and one diode at peak point (V_p, I_p). Thus

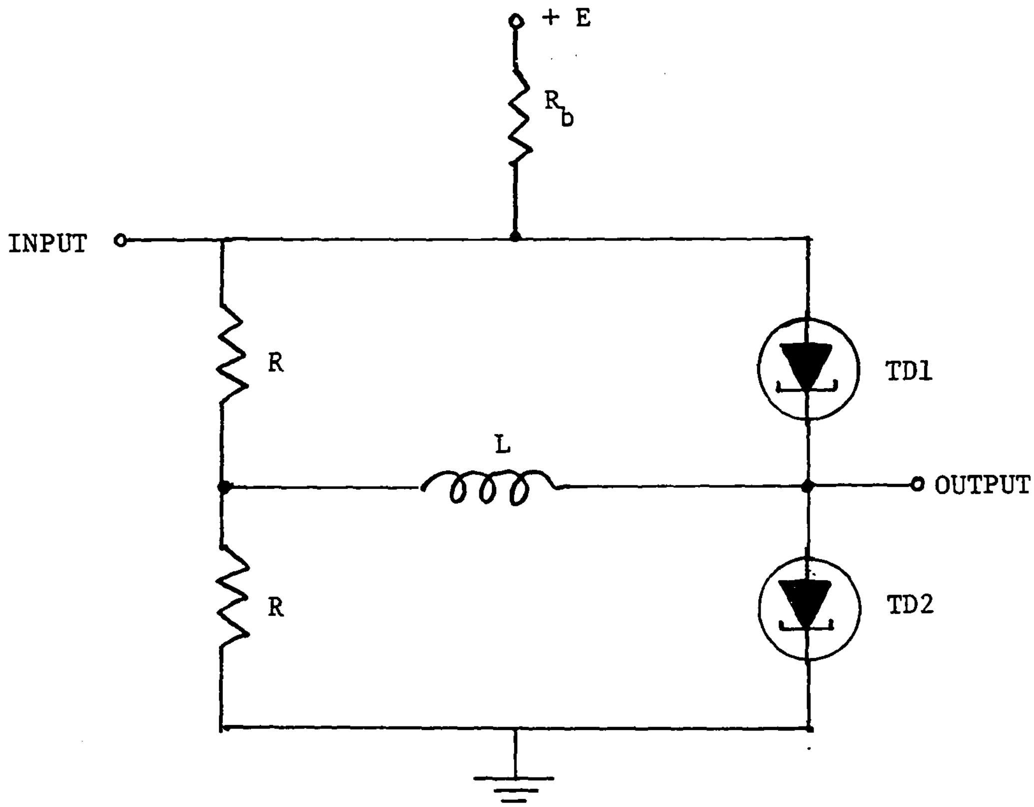


Figure 16a. A two-tunnel-diode flip-flop circuit

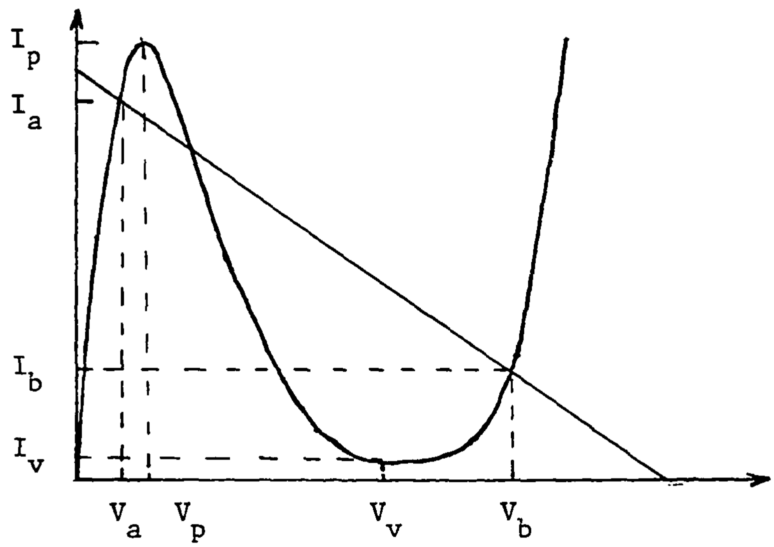


Figure 16b. Steady state conditions

$$\left[\left(\frac{V_p}{R} + I_p\right) R_b + 2V_p\right] < E < \left[2\ V_v + \left(I_v + \frac{V_v}{R}\right) R_b\right]$$

or

$$\left[V_v + V_p + \left(I_p + \frac{V_p}{R}\right)\ R_b\right]$$

whichever is smaller.

The following two loop equations also have to be satisfied.

$$(I_a - I_b)R = V_b - \frac{V_p}{I_p} I_a$$

$$E = R_b(I_a + \frac{V_a}{R}) + V_a + V_b \qquad (7\text{-}10)$$

After algebraic manipulations a suitable set of values for R, R_b and E are:

$R = 0.6056$ kΩ

$R_b = 7.4933$ kΩ

$E = 10$ V

The operating points are obtained with the help of computer by using subroutine ZSYSTM. They are:

$V_a = 0.1356$ V

$I_a = 0.9989$ mA

$V_b = 0.70018$ V

$I_b = 0.0667$ mA

G. Example 7

In Example 1 the v-i characteristics of a tunnel diode were approximated by five piecewise-cubic segments. This example investigates the effects of varying the number of segments. In particular, a three segment approximation with knots (0.0000 V, -0.0579 mA), (0.13808 V, 0.9300 mA), (0.66014 V, -0.0029 mA) and (1.0000 V, 0.8387 mA) as is shown in Figure 17 and a seven segment approximation with knots (0.0000 V, -2.491×10^{-7} mA), (0.09411 V, 0.9275 mA), (0.1508 V, 0.9949 mA), (0.2145 V, 0.8594 mA), (0.3742 V, 0.3094 mA), (0.6220 V, 0.07052 mA), (0.7969 V, 0.08667 mA) and (1.0000 V, 1.0000 mA) are used. The Y vector and spline coefficient matrix for the three segment approximation are

$$Y = \begin{bmatrix} -0.05799 \\ 0.93000 \\ -0.00293 \end{bmatrix}$$

$$C = \begin{bmatrix} 24.920 & -191.80 & 457.300 \\ -1.888 & -2.35 & 4.873 \\ -0.3581 & 5.281 & 9.002 \end{bmatrix}$$

and for the seven segment approximation are

$$Y = \begin{bmatrix} -2.491 \times 10^{-7} \\ 0.9275 \\ 0.9949 \\ 0.8594 \\ 0.3094 \\ 0.07052 \\ 0.08667 \end{bmatrix}$$

$$C = \begin{bmatrix} 17.660 & -99.84 & 177.9 \\ 3.5700 & -49.45 & 129.8 \\ -0.7666 & -27.51 & 97.69 \\ -3.0840 & -8.828 & 41.17 \\ -2.7540 & 10.90 & -14.83 \\ -0.08512 & -0.1286 & 6.535 \\ 0.4696 & 3.3000 & 80.66 \end{bmatrix}$$

Table 3 gives the solutions obtained for the circuit of Example 1 with these approximations and for comparison those obtained with the five piecewise-cubic segment approximation and the original fifth degree polynomial [15]. Comparing these results, it is seen that there is substantial loss in accuracy in going from the five segment approximation to the three segment approximation. While the results of the three segment approximation may be sufficiently accurate for some engineering applications, they are not generally acceptable. The results of the five segment approximation are generally acceptable for engineering applications.

These conclusions are reinforced by Figures 17 and 18. Figure 17 compares the three segment piecewise-cubic approximation with the original v-i characteristics and shows that there are substantial differences. Figure 18 shows that the five segment piecewise-cubic approximation and the original v-i characteristic are almost coincident. This indicates that little improvement would be obtained in going to a seven segment approximation. This conjecture is confirmed by the results obtained with the seven segment approximation as shown in Table 3.

Table 3. Comparison of various approximations

	Fifth degree polynomial	3 segment piecewise-cubic	5 segment piecewise-cubic	7 segment piecewise-cubic
Q1	0.06263 V 0.75824 mA	0.0501 V 0.7666 mA	0.0625 V 0.7582 mA	0.06265 V 0.75823 mA
Q2	0.28537 V 0.60975 mA	0.2898 V 0.6067 mA	0.2864 V 0.6090 mA	0.28585 V 0.60943 mA
Q3	0.88443 V 0.21038 mA	0.8650 V 0.2233 mA	0.8855 V 0.2096 mA	0.88515 V 0.2099 mA

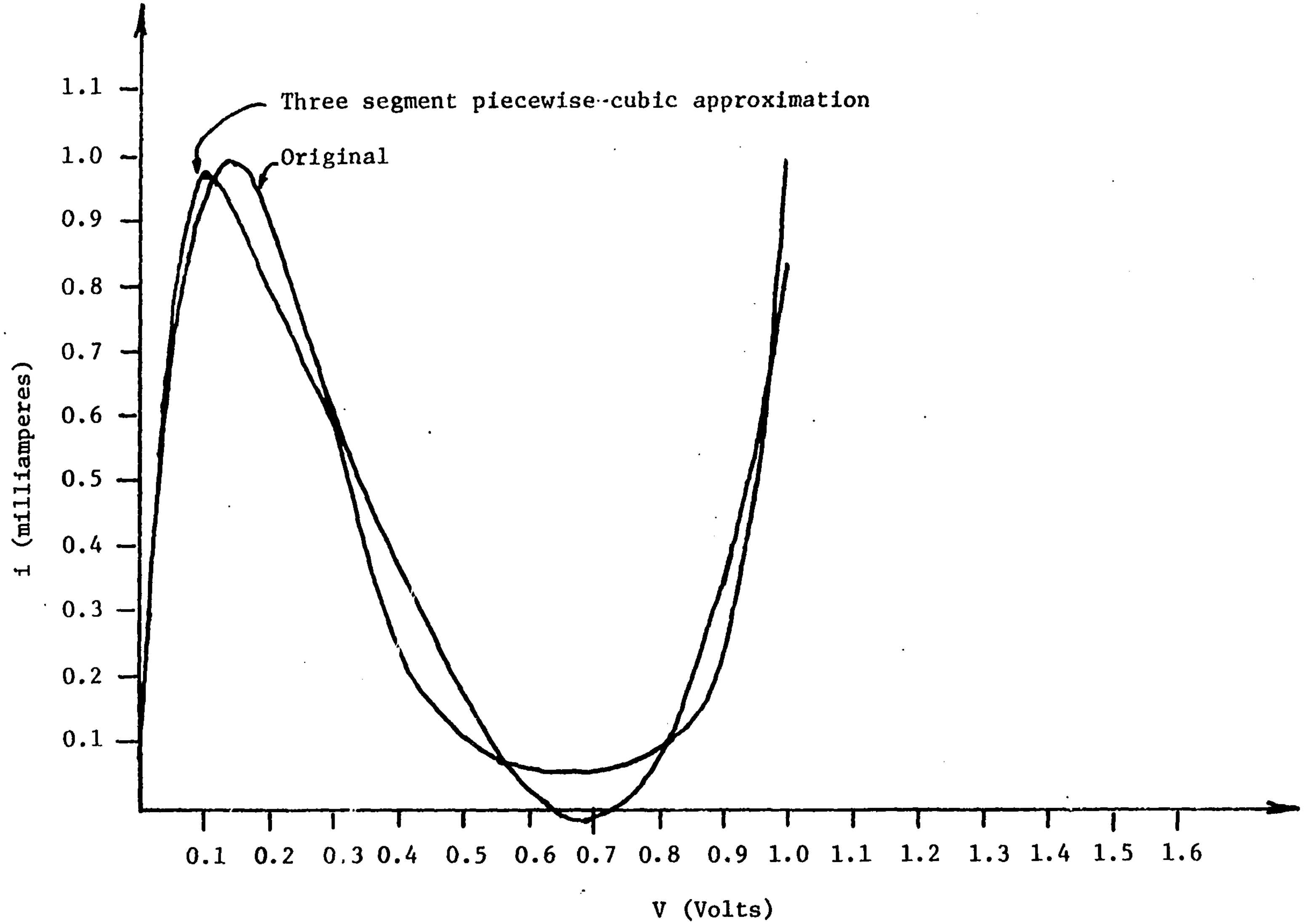


Figure 17. Three segment piecewise-cubic approximation

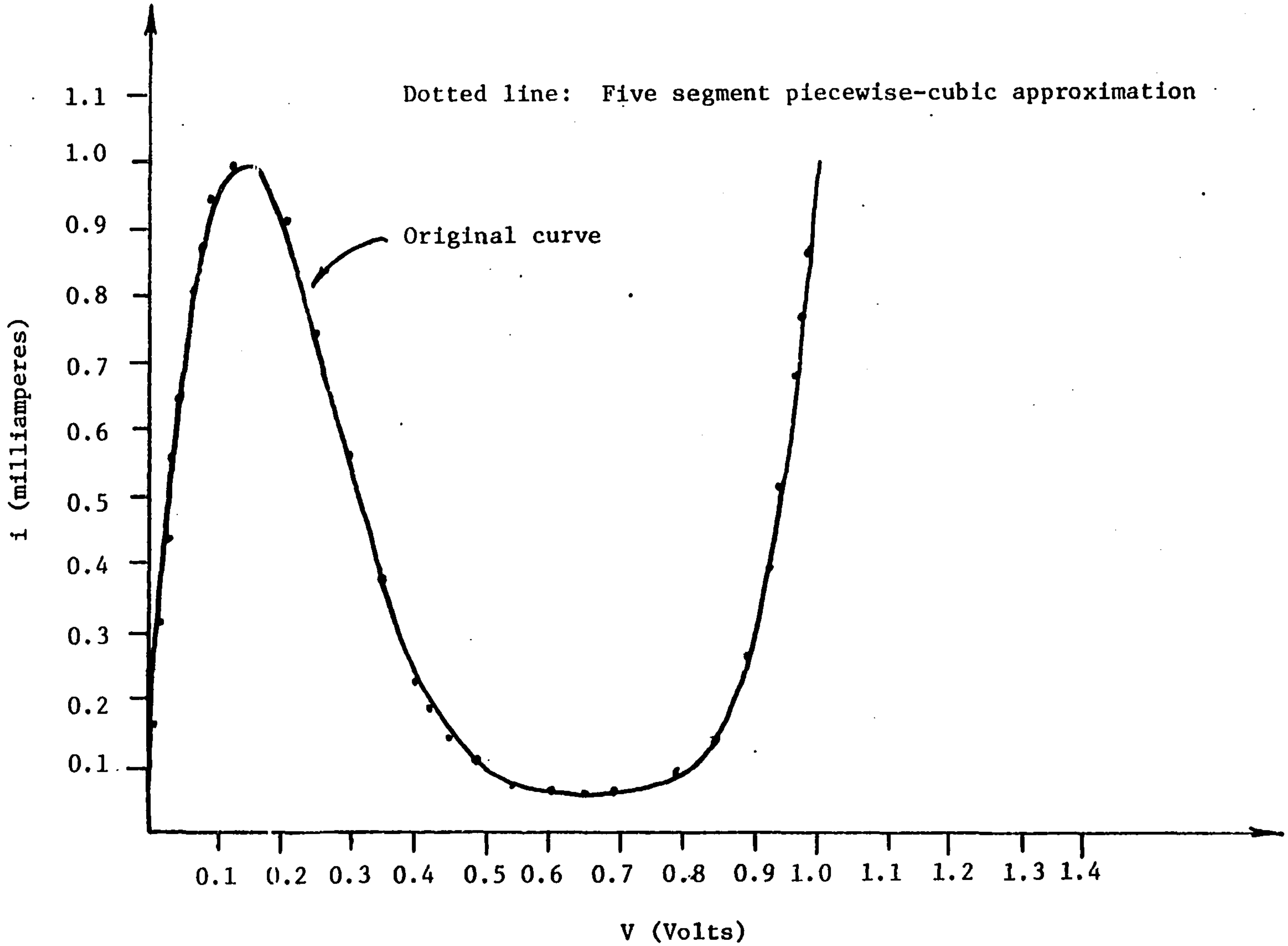


Figure 18. Five segment piecewise-cubic approximation

H. Example 8

This example compares the results obtained with the piecewise-cubic spline technique to those obtained with piecewise-linear methods. The circuit of Example 1 is again used and the v-i characteristics of the tunnel diode are first approximated by five piecewise-linear segments and then by ten piecewise-linear segments as shown in Figure 19 and Figure 20 respectively. The equations for all the segments are given in Table 4. The solutions are obtained by solving the simultaneous linear equations. They are listed in Table 5 for both the five and ten segment approximations. In Table 5 the actual solutions and the solutions obtained by the piecewise-cubic spline technique with five segments are also listed for the purpose of comparison. In all the cases the solutions are obtained with three different applied voltages. The criteria used in comparison is the distance measured between the true solution point and the solution points obtained by the various approximation methods. A smaller distance implies smaller error.

From the table it is observed that, out of nine solutions, only one error by the five-segment piecewise-linear method is less than those by the piecewise-cubic method. Similarly, a comparison of the five-segment cubic approximation with the ten-segment piecewise-linear approximation shows that the former gives more accurate solutions at six of nine points.

This comparison indicates that with nonlinear resistive networks having a negative slope region, the piecewise-cubic technique can be used with fewer segments than the piecewise-linear method and still obtain acceptable accuracy. If there are two nonlinear elements in the circuit as used in Example 3, for a five segment piecewise-cubic approximation there are 20 coefficients, 5 intervals, 6 knots and 25 segment combinations, but for a ten segment piecewise-linear approximation there are 20 coefficients, 10 intervals, 11 knots and 100 segment combinations. Therefore, a great amount of computer storage and time may be saved by using piecewise-cubic spline techniques for analysis of large scale nonlinear networks.

Table 4. Equations for five segment and 10 segment piecewise-linear approximations

5 segment:		
1.	$i = 7.1407\ v$	$0 \leq v \leq 0.14$
2.	$i = -2.8916\ v + 1.4045$	$0.14 \leq v \leq 0.38$
3.	$i = -0.8632\ v + 0.6336$	$0.38 \leq v \leq 0.655$
4.	$i = 0.7861\ v - 0.4465$	$0.655 \leq v \leq 0.9$
5.	$i = 7.391\ v - 6.391$	$0.9 \leq v \leq 1.0$
10 segments:		
1.	$i = 9.459\ v$	$0.0 \leq v \leq 0.1$
2.	$i = -0.439\ v + 0.9898$	$0.1 \leq v \leq 0.2$
3.	$i = -3.451\ v + 1.5922$	$0.2 \leq v \leq 0.3$
4.	$i = -2.999\ v + 1.4566$	$0.3 \leq v \leq 0.4$
5.	$i = -1.5\ v + 0.8568$	$0.4 \leq v \leq 0.5$
6.	$i = -0.37\ v + 0.2918$	$0.5 \leq v \leq 0.6$
7.	$i = -0.0146\ v + 0.07856$	$0.6 \leq v \leq 0.7$
8.	$i = 0.1629\ v - 0.0456$	$0.7 \leq v \leq 0.8$
9.	$i = 1.7627\ v - 1.3255$	$0.8 \leq v \leq 0.9$
10.	$i = 7.391\ v - 6.391$	$0.9 \leq v \leq 1.0$

Table 5. Error comparison for various approximations

E(volts)		ACTUAL SOLUTIONS	PWC SOLUTION	ERROR	PWL 5 SEG.	ERROR	PWL 10 SEG	ERROR
	Q1	0.04135	0.0414	0.000139	0.0768	0.04265	0.0592	0.0218
		0.57243	0.5723		0.5487		0.5599	
0.9	Q2	0.36030	0.3520	0.00995	0.3615	0.00138*	0.3672	0.00823*
		0.35980	0.3653		0.3591		0.3553	
	Q3	0.78376	0.7775	0.00748	0.7203	0.07621	0.7783	0.0065*
		0.07749	0.0816		0.1197		0.0811	
	Q1	0.06263	0.0625	0.000136	0.1024	0.04792	0.0790	0.0197
		0.75824	0.7582		0.7315		0.7472	
1.2	Q2	0.28537	0.2864	0.00127	0.2716	0.0166	0.2846	0.000809*
		0.60975	0.6090		0.6191		0.6100	
	Q3	0.88443	0.8855	0.00132	0.8580	0.0317	0.8749	0.01138
		0.21038	0.2096		0.2379		0.2166	
	Q1	0.09675	0.0962	0.00062	0.1280	0.03765	0.0987	0.00240
		0.93550	0.9358		0.9145		0.9341	
1.5	Q2	0.21540	0.2168	0.00244	0.1817	0.0405	0.2127	0.00319
		0.85640	0.8544		0.8790		0.8581	
	Q3	0.92725	0.9276	0.000418	0.9172	0.01179	0.9172	0.01179
		0.38183	0.3816		0.3880		0.3800	

*Indicates the error is less than piecewise-cubic method

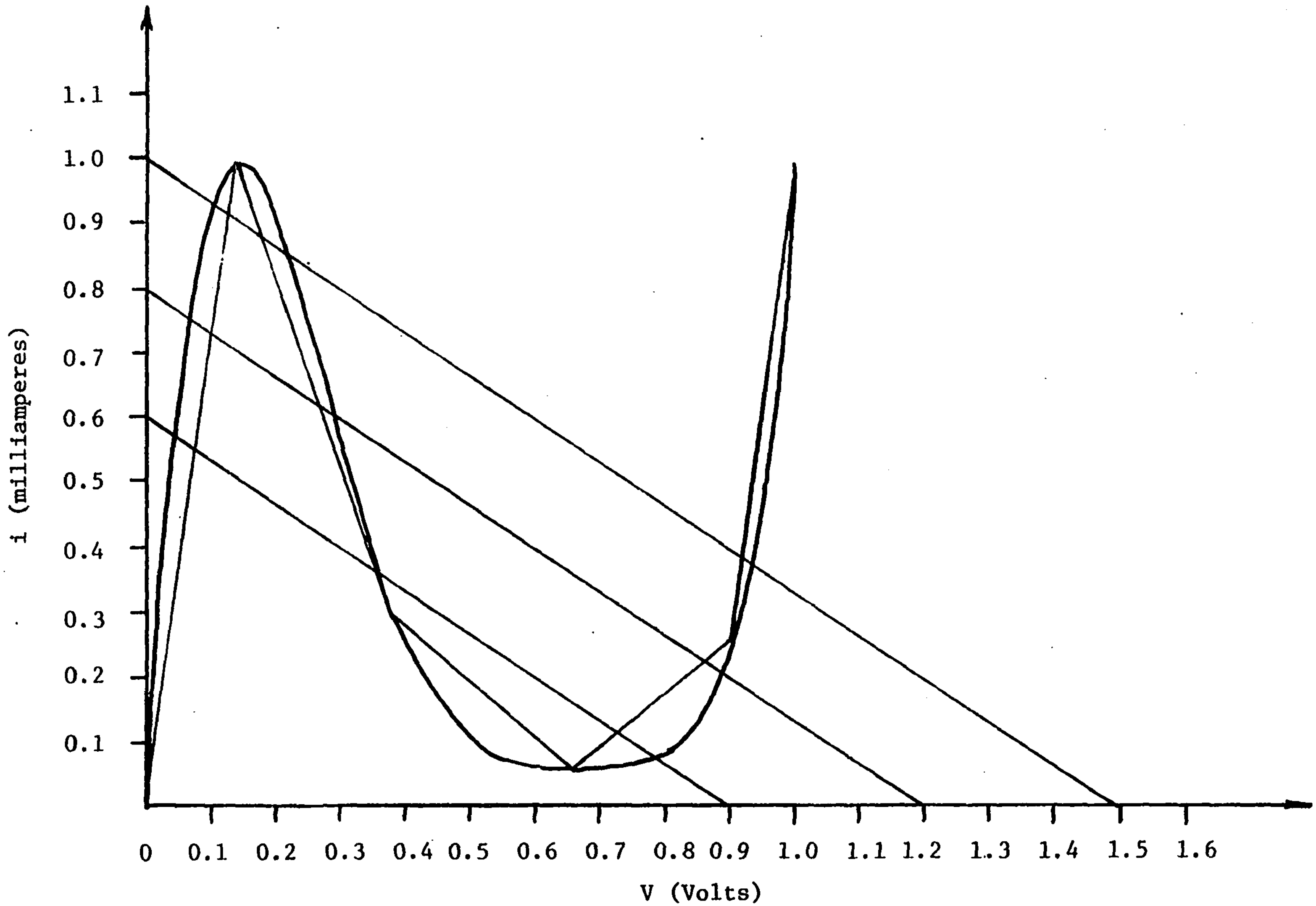


Figure 19. Five segment piecewise-linear approximation

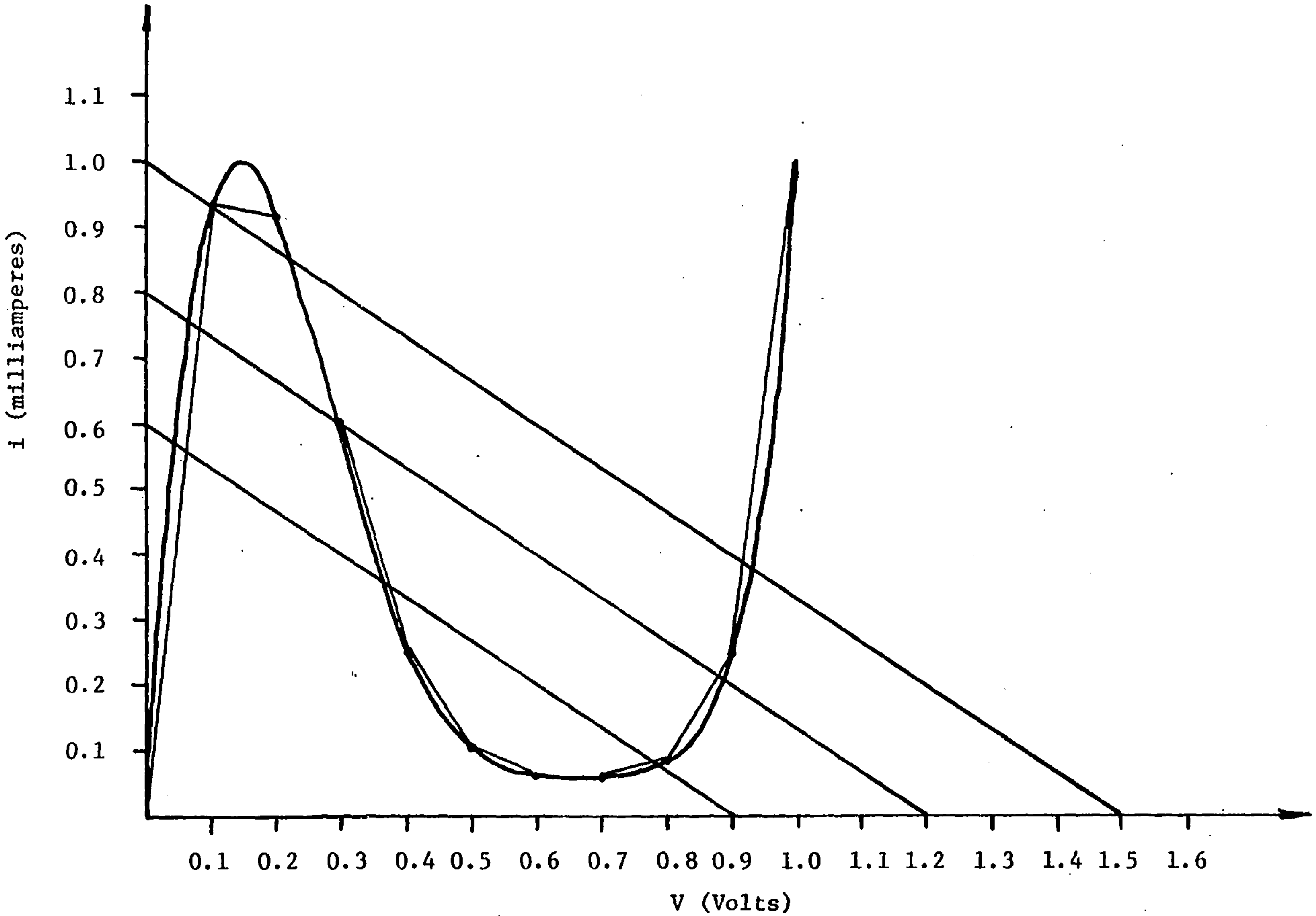


Figure 20. Ten segment piecewise-linear approximation

VIII. SUMMARY AND SUGGESTED CONSIDERATION

Spline functions are well-developed in the literature and have been used by engineers in the fields of control theory, system modeling and identification, communication and digital filtering [20] because of their existence and uniqueness, minimum norm and good approximation properties [18].

The method of using piecewise-cubic spline approximations is superior to other techniques when the nonlinear element itself has one or more regions of short radii of curvature in its v-i curve. Spline functions allow the condition of continuous first and/or second derivatives at the knots, or junctions of succeeding spline segments. In contrast, piecewise-linear models are superior when the v-i curves have sections meeting "nearly at a point", as, for example, when saturation exists. Specific cases in which the piecewise-cubic spline technique is advantageous are described in the following paragraphs.

Piecewise-cubic spline approximation should be applied when the v-i characteristics of the nonlinear elements are represented by high degree polynomials or analytic functions, like the one used in the tunnel diode circuit of Example 1. The number of knots to be used and their initial locations depend on the shape of the v-i curve. As a guideline and the starting point, the number of knots can be taken as twice the number of relative maximum and minimum points. For approximating a highly non-linear v-i curve with very few knots the fixed knots algorithm should be used in order to avoid sharp bend in the subinterval and miss some of the solutions. Some knot locations be selected near the relative maximum and minimum points.

Piecewise-cubic approximation should be applied when the v-i characteristics are represented by smooth experimental curve with high non-linearity, like the one used in Example 4. Since many of the v-i characteristics of the electronic devices are represented by experimental curves, this technique is suitable for this purpose.

In analyzing large scale nonlinear networks by computer, it is a common practice to reduce the number of variables by combining nonlinear elements in the network in order to perform the analysis. In the process of combining nonlinear elements together, v-i characteristics with high nonlinearity may appear. In this case the piecewise-cubic spline technique should be applied. The circuit used in Example 3 demonstrates this process when two tunnel diodes are connected in series. When they are combined together a smooth highly nonlinear v-i curve appear as is shown in Figure 12. In the analysis of large scale electronic circuit this situation often happens.

Nonlinear solid state devices such as diode and transistors can be modeled by spline functions [20]. To analyze the networks consisting of these devices the piecewise spline technique can easily be applied. The tunnel diode circuit of Example 1 is used in digital computer and control areas where extremely fast logic gates, registers and memories are needed. The tunnel diode flip-flop circuit used in Example 6 is a basic computer logic circuit. The tunnel diode detector circuit of the fire-warning and fire-extinguishing actuator for an aircraft jet engine is another important application [2]. Other areas where tunnel diodes are used are electronic amplifier and oscillator circuits and the controlled-negative resistive device. If a large number of tunnel diode are used they can even provide step approximations of desired graphical function [27].

In the transmission of data over a communication channel, the piece-spline technique can be used for data compression. Such data compression schemes can be employed in the compression of speech signals, picture functions [20] and electroencephalogram (EEG) signals. Cubic splines can also be used in the design of digital filters [20].

In Example 7, the approximation property and the significance of piecewise-cubic spline approximants is demonstrated. Five piecewise-cubic segments are needed to accurately approximate the fifth degree polynomial. Comparison of the solutions obtained for the circuit of Example 1 by three segment, five segment and seven segment piecewise-cubic approximations shows that the solutions with the five segment approximation are generally acceptable for engineering applications.

In Example 8 a comparison is made between piecewise-cubic spline and piecewise-linear techniques in analyzing the nonlinear resistive networks. It is shown that the piecewise-cubic spline technique with five segment approximation is more accurate than the piecewise-linear method with ten segment approximation. For two nonlinear resistors as were used in Example 3 the ratio of segment combinations are 4:1 ($(100/25)^2$). In general, for analyzing a large scale nonlinear resistive networks with n identical elements which are characterized by highly nonlinear smooth curves the ratio of segment combinations would be (N_ℓ/N_c) where N_ℓ and N_c denote the required number of piecewise-linear segments and piecewise-cubic segments respectively for approximating the v-i curve. If N_ℓ/N_c is greater than two a great amount of computer storage and time may be saved.

As for approximating functions of several variables there is still no computationally efficient techniques available [32, 33]. The future hopes rest on the more efficient n-dimensional generalization of the promising spline function approach [33]. It is hoped that this study will be a stepping stone to apply spline function techniques in analyzing nonlinear electronic network problems.

For a given nonlinear resistive network with multiple solutions, determination of the number of solutions which the network has is still an open and difficult problem because different bias conditions in the circuit result in different number of solutions. Chien [34] investigated this problem by establishing conditions that a bounded set contains an operating point. The conditions show that close relations exist between eventual passivity and the existence of an operating point in a bounded set. But his method for determining the existence of an operating point is limited to nonlinear resistors with very simple v-i curves. Moreover he still has to apply the piecewise-linear method to compute the operating point. Therefore more work is needed to develop a general practical theory to estimate the number of true solutions before actually computing the solutions in order to save computer time. This work has shown that piecewise-cubic approximants can be used to find, in an efficient manner, the actual solutions when multiple solutions are present. A combination of knowledge of the number of solution and cubic approximants would give the circuit designer a powerful technique for the analysis of large scale electronic circuits.

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X. ACKNOWLEDGMENTS

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XI. APPENDIX A: SUBROUTINE ICSFKU

1. This subroutine [29] computes a least square approximation to a given set of points using cubic splines with a given set of knots. The successful use of splines for purposes of providing a smooth approximation to a given set of points depends strongly on the proper placement of knots.
2. ICSFKU is intended to be used for functions that can be approximated adequately with relatively few knots. It has a built-in limit of 28 knots.
3. If the knots are ordered, then interval I has endpoints XK(I) and XK(I + 1). The cubic spline function is given by

   S(T) = ((C(I, 3)*D + C(I, 2)) * D + C(I, 1)) * D + Y(I)

   where T is in interval I and D = T - XK (I).

   The spline coefficients are always computed as if the knots are ordered, and ordering the knots is advised for ease of output usage.
4. The cubic spline computed by ICSFKU is continuous and has continuous first and second derivatives.
5. The error which ICSFKU minimize is defined as follows:

$$\text{ERROR} = \sqrt{\sum_{i=1}^{N_x} R_i\, W_i} \qquad \text{where}$$

$$R_i = F_i - S(X_i) \qquad i = 1, \ldots, N_x$$

$$W_1 = (X_2 - X_1)/(X_{Nx} - X_1)$$

$$W_i = (X_{i+1} - X_{i-1})/(X_{Nx} - X_1) \qquad i = 2, \ldots, Nx-1$$

$$W_{Nx} = (X_{Nx} - X_{Nx-1})/(X_{Nx} - X_1)$$

(X_i, F_i) $i = 1, \ldots, Nx$ is the given set of points and S is the least squares cubic spline approximation to that set of points.

XII. APPENDIX B: SUBROUTINE ICSVKU

1. This subroutine [29] computes a least squares approximation to a given set of points by cubic splines with a given number of knots. The successful use of splines for purposes of providing a smooth approximation to a given set of points depends strongly on the proper placement of knots. ICSVKU starts with a given set of knots and varies them one by one in order to determine the knot locations that minimize the least squares error.
2. ICSVKU uses the fixed knot subroutine ICSFKU in order to determine the optimal knot locations. Each knot is, in turn, varied so as to minimize the least squares error as a function of this knot. This process is started with the right most interior knot and proceeds sequentially to the left. Iterations continue until a termination criterion is met.
3. The cubic spline computed by ICSVKU is continuous and has continuous first and second derivatives. The number of knots must not be greater than 28.

XIII. APPENDIX C: SUBROUTINE ZSYSTM

1. ZSYSTM [29] solves a system of N simultaneous nonlinear equations in N unknowns.
2. This subroutine uses Brown's method [35] which is at least quadratically convergent and requires only $N^2/2 + 3N/2$ function evaluations per iterative step as compared with $N^2 + N$ evaluations for Newton's method. A root is accepted if two successive approximations to a given root agree in the first NSIG digits. A root is also accepted if $|F(X, K, PAR)|$ is less than EPS for every K = 1, ..., N.
3. ZSYSTM will terminate processing if a root is not found within ITMAX iterations and/or if the Jacobian matrix of the system of equations becomes computationally singular. In this case, a different initial approximation should be tried and/or the equations should be studied to see if some of the equations or variables can be eliminated or solved for in terms of others.

XIV. APPENDIX D: COMPUTER PROGRAM LISTINGS

All computer programs listed here were written to conform to IBM FORTRAN IV, level G language rules. All programs and subroutines were written in double precision arithmetic.

```
C.......................................................................
C
C      MAIN PROGRAM
C
C
C      PURPOSE:
C         INTERPOLATION OF FIFTH ORDER TUNNEL DIODE V-I
C      CHARACTERISTICS BY SPLINE FUNCTION
C
C      METHOD:
C          SUBROUTINE ICSVKU COMPUTES LEAST SQUARES APPROXIMATION
C      TO A GIVEN SET OF POINTS USING CUBIC SPLINES WITH A GIVEN
C      SET OF KNOTS
C
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION X(22),F(22),XK(6),Y(5),C(5,3),WK(264),FK(6),
      1VIC(5),CIC(5)
       IC=5
       NXK=6
C
C      INITIAL KNOTS
       XK(1)=0.0
       XK(2)=1.3808D-1
       XK(3)=0.5D0
       XK(4)=6.6014D-1
       XK(5)=0.85D0
       XK(6)=1.0D0
C
       NX=22
C      READ IN THE SELECTED NX POINTS OF X(I)
       READ (5,200) (X(I), I=1,NX)
 200   FORMAT(6D10.4)
C      CALCULATE F(I) CORRESPONDING TO X(I)
       DO 5 I=1,NX
```

```
      F(I)=((((8.372D1*X(I)-2.2631D2)*X(I)+2.2962D2)*X(I)
     1-1.0379D2)*X(I)+1.776D1)*X(I)
 5    CONTINUE
      WRITE (6,150) (X(I),F(I),I=1,22)
 150  FORMAT ('0', 2D20.8)
C
C
      CALL ICSVKU(X,F,NX,XK,NXK,Y,C,IC,ERROR,WK,IER)
C     PRINT  OUT THE CONSTANT TERM OF THE COEFFICIENT MATRIX C(I,J)
      WRITE (6,10)
 10   FORMAT('0',10X,'Y')
      WRITE (6,30) (Y(I),I=1,5)
 30   FORMAT('0',5X,D10.4)
C     PRINT OUT THE COEFFICIENTS MATRIX C(I,J) EXCEPT THE CONSTANT
C     TERM
      WRITE(6,40)
 40   FORMAT('0',40X,'C')
      WRITE (6,50) ((C(I,J),J=1,3),I=1,5)
 50   FORMAT('0',3X,3D20.4)
C     PRINT OUT THE OPTIMAL KNOTS CALCULATED
      WRITE(6,70) XK
 70   FORMAT('0',10X,6D15.4)
      WRITE(6,80) IER,ERROR
 80   FORMAT('0',10X,I4,D20.4)
C     CALCULATE THE CORRESPONDING F(I) FOR OPTIMAL KNOTS
      DO 100 I=1,NXK
      FK(I)=((((8.372D1*XK(I)-2.2631D2)*XK(I)+2.2962D2)*XK(I)
     1-1.0379D2)*XK(I)+1.776D1)*XK(I)
 100  CONTINUE
      WRITE (6,110)  (FK(I),I=1,NXK)
 110  FORMAT('0',10X,6D15.4)
C     CALCULATE THE INITIAL CONDITIONS FOR EACH INTERVAL
      DO 120 I=1,IC
      J=I+1
      VIC(I)=(XK(I)+XK(J))/2.0D0
```

```
      CIC(I)=(FK(I)+FK(J))/2.0D0
  120 CONTINUE
      WRITE(6,130) (VIC(I),I=1,IC)
  130 FORMAT('0',10X,5D15.4)
      WRITE(6,130) (CIC(I),I=1,IC)
      STOP
      END
C.....................................................................
```

```
C......................................................................
C
C     MAIN PROGRAM
C
C
C     PURPOSE:
C        FINDING THE OPERATING POINTS AND THE THRESHOLD LEVEL FOR
C     TUNNEL DIODE CIRCUIT
C
C     METHOD:
C        THE V-I CHARACTERISTICS OF THE NONLINEAR RESISTOR IS
C     SUBSTITUTED BY ITS EQUIVALENT SPLINE SEGMENT. THE VALUE OF E
C     IS CANGED FROM 0.0 TO 2.0
C        USE SUBROUTINE ZSYSTM TO SOLVE THE SYSTEM OF NONLINEAR
C     NETWORK EQUATIONS. THE FINAL VALID SOLUTIONS ARE OBTAINED
C     BY VERIFICATION
C
C     LISTING OF SIGNIFICANT VARIABLE NAMES
C     XVL,XVH--LOWER AND UPPER LIMITS OF THE VOLTAGE IN EACH INTERVAL
C     XIL,XIH--LOWER AND UPPER LIMITS OF THE CURRENT IN EACH INTERVAL
C
C
      EXTERNAL           AUX
      DIMENSION X(2),X1(5),X2(5),WA(12),IPAR(2),XK(6),FK(6),XVL(6),
     1 XVH(6),XIL(6),XIH(6)
      DOUBLE PRECISION X,X1,X2,WA,EPS,E
      COMMON/EX/E
C
C     READ IN INITIAL CONDITIONS
      READ(5,50) (X1(I), I=1,5)
 50   FORMAT(5D10.4)
      READ(5,50) (X2(I), I=1,5)
C
C     READ IN OPTIMAL KNOTS XK(I)
      READ (5,70) (XK(I),I=1,6)
```

```
 70    FORMAT (6D10.4)
       READ(5,80) (XIL(I),I=1,5)
 80    FORMAT (5D10.4)
       READ(5,80)  (XIH(I),I=1,5)
       DO 200  I=1,5
       XVL(I)=XK(I)
       J=I+1
       XVH(I)=XK(J)
 200   CONTINUE
       EPS=1.0D-5
       NSIG=4
       N=2
       ITMAX=300
       E=0.0D0
       DO 20  J=1,20
       E=E+0.1D0
       DO 60  I=1,5
       IPAR(1)=I
       X(1)=X1(I)
       X(2)=X2(I)
       CALL ZSYSTM (AUX,EPS,NSIG,N,X,ITMAX,WA,IPAR,IER)
C      VERIFY AND FIND OUT THE TRUE SOLUTIONS
       IF(X(1)  .LT. XVL(I) .OR. X(1) .GT. XVH(I)) GO TO 60
       IF(X(2) .LT. XIL(I)  .OR. X(2) .GT. XIH(I)) GO TO 60
       IF(X(1) .GT. XVL(I)  .AND. X(1) .LT. XVH(I)) GO TO  90
       IF(X(2) .GT. XIL(I)  .AND. X(2) .LT. XIH(I)) GO TO 90
 90    WRITE (6,1000) IER,ITMAX,I,X
 1000  FORMAT(1H , 2I4,10X,'TS(',I3,')=',2D20.8)
 60    CONTINUE
 20    CONTINUE
       STOP
       END
       FUNCTION AUX(X,M,IPAR)
       DIMENSION X(1),IPAR(2)
       COMMON/EX/E
```

```
C      INDICATE 5 SPLINE SEGMENTS
       INTEGER II(5)/1,2,3,4,5/
       DOUBLE PRECISION  AUX,X,E
       I=IPAR(1)
       GO TO (1,2),M
 1     N=II(I)
       GO TO 4
 2     N=6
 4     GO TO (11,12,13,14,15,16),N
 11    AUX=X(2)-((1.522D2*X(1)-9.517D1)*X(1)+1.747D1)*X(1)-4.074D-4
       RETURN
 12    AUX=X(2)-((4.592D1*(X(1)-1.711D-1)-1.706D1)*(X(1)-1.711D-1)
      1-1.734D0)*(X(1)-1.711D-1)-0.9655D0
       RETURN
 13    AUX=X(2)-((-5.237D1*(X(1)-4.075D-1)+1.55D1)*(X(1)-4.075D-1)
      1-2.101D0)*(X(1)-4.075D-1)-0.209D0
       RETURN
 14    AUX=X(2)-((-1.679D0*(X(1)-0.4919D0)+2.237D0)*(X(1)-0.4919D0)
      1-0.6034D0)*(X(1)-0.4919D0)-0.1106D0
       RETURN
 15    AUX=X(2)-((7.59D1*(X(1)-0.7796D0)+0.7874D0)*(X(1)-0.7796D0)
      1+0.2665D0)*(X(1)-0.7796D0)-0.08211
       RETURN
 16    AUX=X(1)+1.5D0*X(2)-E
       RETURN
       END
C.......................................................................
```

```
C•••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••••
C
C     MAIN PROGRAM
C
C
C     PURPOSE:
C        FIND THE OPERATING POINTS FOR THE NONLINEAR RESISTIVE
C     NETWORK WIRH ONE TUNNEL DIODE,ONE UNICURSAL ELEMENT WITH
C     SELF-INTERSECTIONS,ONE CURRENT-CONTROLLED VOLTAGE
C     SOURCE AND TWO LINEAR RESISTORS
C
C     METHOD:
C        AFTER FORMULATING HYBRID EQUATIONS OF THE NETWORK, USE
C     SUBROUTINE ZSYSTM TO OBTAIN THE TRUE SOLUTIONS
C
C     LISTING THE SIGNIFICANT VARIABLE NAMES
C     X1-VOLTAGE OF R1
C     X2-VOLTAGE OF R2
C     X3-CURRENT OF R1
C     X4-CURRENT OF R2
C     X1VK-KNOTS FOR R1
C     X1VL,X1VH-LOWER AND UPPER LIMITS OF X1 IN EACH INTERVAL
C     X2VL,X2VH-LOWER AND UPPER LIMITS OF X2 IN EACH INTERVAL
C     X3IL,X3IH-LOWER AND UPPER LIMITS OF X3 IN EACH INTERVAL
C     X4IL,X4IH-LOWER AND UPPER LIMITS OF X4 IN EACH INTERVAL
C
C
      EXTERNAL        AUX
      DIMENSION X(4),X1(5),X2(9),X3(5),X4(9),WA(21),IPAR(2),
     1X1VK(6),X1VL(5),X1VH(5),X3IL(5),X3IH(5),X4IL(9),
     1X4IH(9),X2VL(9),X2VH(9)
      DOUBLE PRECISION X,X1,X2,X3,X4,WA,EPS,X1VK,X1VL,X1VH,
     1X3IL,X3IH,X4IL,X4IH,X2VL,X2VH
C
C     READ IN INITIAL CONDITIONS
```

```
      READ(5,50) (X1(I), I=1,5)
 50   FORMAT(5D10.4)
      READ(5,50) (X2(I), I=1,9)
      READ(5,50) (X3(I),I=1,5)
      READ(5,50) (X4(I),I=1,9)
C
C     READ IN KNOTS FOR R1
      READ(5,70) (X1VK(I),I=1,6)
 70   FORMAT(6D10.4)
C
      READ(5,50) (X3IL(I),I=1,5)
      READ(5,50) (X3IH(I),I=1,5)
C     READ IN LOWER AND UPPER LIMITS OF EACH INTERVAL FOR R2
      READ(5,80) (X4IL(I),I=1,9)
 80   FORMAT(5D10.4)
      READ(5,80) (X4IH(I),I=1,9)
      READ(5,80) (X2VL(I),I=1,9)
      READ(5,80) (X2VH(I),I=1,9)
C
C     CALCULATE X1VL AND X1VH
      DO 200 I=1,5
      J=I+1
      X1VL(I)=X1VK(I)
      X1VH(I)=X1VK(J)
 200  CONTINUE
      WRITE(6,100) (X1(I), I=1,5)
 100  FORMAT (1H , 5D20.8)
      WRITE(6,100) (X3(I), I=1,5)
      WRITE(6,100) (X2(I), I=1,9)
      WRITE(6,100) (X4(I), I=1,9)
      WRITE (6,300) (X1VK(I),I=1,6)
 300  FORMAT(1H , 6D20.8)
      WRITE(6,100) (X1VL(I),I=1,5)
      WRITE(6,100) (X1VH(I),I=1,5)
      WRITE(6,100) (X3IL(I),I=1,5)
```

```
      WRITE(6,100) (X3IH(I),I=1,5)
      WRITE(6,100) (X2VL(I),I=1,9)
      WRITE(6,100) (X2VH(I),I=1,9)
      WRITE(6,100) (X4IL(I),I=1,9)
      WRITE(6,100) (X4IH(I),I=1,9)
      EPS=1.0D-5
      NSIG=4
      N=4
      ITMAX=300
C
C     START COMPUTING
      DO 60 I=1,5
      DO 60 J=1,9
      IPAR(1)=I
      IPAR(2)=J
      X(1)=X1(I)
      X(2)=X2(J)
      X(3)=X3(I)
      X(4)=X4(J)
C
      CALL ZSYSTM (AUX,EPS,NSIG,N,X,ITMAX,WA,IPAR,IER)
C     CHECKING THE SOLUTIONS WHETHER THEY FALL WITHIN
C     THE INTERVAL LIMITS
      IF(X(1) .LT. X1VL(I) .OR. X(1) .GT. X1VH(I)) GO TO 60
      IF (X(2) .LT. X2VL(J) .OR. X(2) .GT. X2VH(J)) GO TO 60
      IF(X(3) .LT. X3IL(I) .OR. X(3).GT. X3IH(I)) GO TO 60
      IF(X(4) .LT. X4IL(I) .OR. X(4) .GT. X4IH(I)) GO TO 60
      IF(X(1) .GT. X1VL(I) .AND. X(1) .LT. X1VH(I)) GO TO 90
      IF(X(2) .GT. X2VL(J) .AND. X(2) .LT. X2VH(J)) GO TO 90
      IF(X(3) .GT. X3IL(I) .AND. X(3) .LT. X3IH(I)) GO TO 90
      IF(X(4) .GT. X4IL(I) .AND. X(4) .LT. X4IH(I)) GO TO 90
C     PRINT OUT THE TRUE SOLUTIONS
 90   WRITE(6,1000) IER,ITMAX,I,J,X
 1000 FORMAT(1H , 2I4,10X,'TS(',2I4,')=',4D20.8)
 60   CONTINUE
```

```
      STOP
      END
      FUNCTION AUX(X,M,IPAR)
      DIMENSION X(1),IPAR(2)
C     INDICATE 5 SPLINE SEGMENTS FOR R1
      INTEGER II(5)/1,2,3,4,5/
C     INDICATE 9 SPLINE SEGMENTS FOR R2
      INTEGER JJ(9)/6,7,8,9,10,11,12,13,14/
      DOUBLE PRECISION  AUX,X
      I=IPAR(1)
      J=IPAR(2)
      GO TO (1,2,3,4),M
 1    N=II(I)
      GO TO 10
 2    N=JJ(J)
      GO TO 10
 3    N=15
      GO TO 10
 4    N=16
      GO TO 10
 10   GO TO (11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26),N
C     11 TO 15 FROM LAW OF ELEMENT OF R1
 11   AUX=X(3)-((1.522D2*X(1)-9.517D1)*X(1)+1.747D1)*X(1)-4.074D-4
      RETURN
 12   AUX=X(3)-((4.592D1*(X(1)-1.711D-1)-1.706D1)*(X(1)-1.711D-1)
     1-1.734D0)*(X(1)-1.711D-1)-0.9655D0
      RETURN
 13   AUX=X(3)-((-5.237D1*(X(1)-4.075D-1)+1.55D1)*(X(1)-4.075D-1)
     1-2.101D0)*(X(1)-4.075D-1)-0.209D0
      RETURN
 14   AUX=X(3)-((-1.679D0*(X(1)-0.4919D0)+2.237D0)*(X(1)-0.4919D0)
     1-0.6034D0)*(X(1)-0.4919D0)-0.1106D0
      RETURN
 15   AUX=X(3)-((7.59D1*(X(1)-0.7796D0)+0.7874D0)*(X(1)-0.7796D0)
     1+0.2665D0)*(X(1)-0.7796D0)-0.08211
```

```
      RETURN
C     16 TO 24 FROM LAW OF ELEMENT OF R2
 16   AUX=X(2)-((9.071D-3*X(1)-0.1913D0)*X(4)+1.966D0)*X(4)-0.9015D-1
      RETURN
 17   AUX=X(2)-((-0.8903D-2*(X(4)-6.983D0)-0.1258D-2)*(X(4)-6.983D0)
     1+0.6213D0)*(X(4)-6.983D0)-7.579D0
      RETURN
 18   AUX=X(2)-((0.1387D-2*(X(4)-1.365D1)-0.1793D0)*(X(4)-1.365D1)
     1-0.5823D0)*(X(4)-1.365D1)-9.028D0
      RETURN
 19   AUX=X(2)-((-0.2575D0*(X(4)-5.2D0)+1.257D0)*(X(4)-5.2D0)
     1-1.974D0)*(X(4)-5.2D0)-3.001D0
      RETURN
 20   AUX=X(2)-((0.4954D-2*(X(4)-6.79D0)+0.2894D-1)*(X(4)-6.79D0)
     1+0.7102D-1)*(X(4)-6.79D0)-2.006D0
      RETURN
 21   AUX=X(2)-((5.31D0*(X(4)-1.402D1)+0.1365D0)*(X(4)-1.402D1)
     1+1.267D0)*(X(4)-1.402D1)-5.91D0
      RETURN
 22   AUX=X(2)-((-0.2973D-1*(X(4)-5.2D0)-0.9309D-1)*(X(4)-5.2D0)
     1+1.4D0)*(X(4)-5.2D0)-3.183D0
      RETURN
 23   AUX=X(2)-((0.2653D0*(X(4)-9.86D0)-0.5087D0)*(X(4)-9.86D0)
     1-1.404D0)*(X(4)-9.86D0)-4.679D0
      RETURN
 24   AUX=X(2)-((-0.1998D-1*(X(4)-1.101D1)+0.4041D0)*(X(4)-1.101D1)
     1-1.524D0)*(X(4)-1.101D1)-2.8D0
      RETURN
C     25 AND 26 FROM LAW OF INTERCONNECTION
 25   AUX=X(3)+0.3D0*X(1)-0.3D0*X(4)-0.36D0
      RETURN
 26   AUX=X(2)+0.9D0*X(1)+0.6D0*X(4)-1.08D0
      RETURN
      END
C.....................................................................
```